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SUBORBITS AND GROUP EXTENSIONS OF FLOWS

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ABSTRACT

Given a pair of an ergodic measured discrete equivalence relation $\mathcal{R}$ and a subrelation $\mathcal{S} \subset \mathcal{R}$ of finite index, a classification of the inclusion up to orbit equivalence will be discussed. In case of amenable and type III_0 relations, the orbit equivalence classes of inclusions will be completely classified in terms of a collection of a subgroup H and a normal subgroup G_0 of a finite group G and an ergodic group (G/G_0) extension of a non-singular flow. This is a generalization of Krieger's theorem by which orbit equivalence classes of single relations were classified. Due to this result, essential type III inclusions will be made clear.

1. Introduction

Given ergodic non-singular transformations R and S of a Lebesgue space, we suppose each S orbit is contained in an R orbit. Our concern is to see how measurably S sits in an R-orbit in view of orbit equivalence. This question has a close connection with a classification of subfactors in von Neumann algebra theory. As a matter of fact, so called group measure space construction factors by R and S, respectively, give us a factor and a subfactor. When we deal with single transformations R, orbit equivalence of R tells us a classification of the corresponding

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factors. But what about the case of pairs of transformations R and S? In this setup it is very natural to consider orbit equivalence preserving suborbits. Our purpose of this article is to classify suborbits up to orbit equivalence.

As we are concerned with orbits, we should adopt pairs of a measured discrete ergodic equivalence relation $\mathcal{R}$ and a subrelation $\mathcal{S}$ instead of R and S. In [Sut1], Sutherland claimed that if the number of $\mathcal{S}$ orbits in an $\mathcal{R}$ orbit is finite then the orbit equivalence of an inclusion $\mathcal{S} \subset \mathcal{R}$ is described in terms of the cross product of a common subrelation $\mathcal{P}$ by a finite group action α_G and a subgroup action α_H respectively. It is shown in [Ham2] that the collection $\{\mathcal{P}, H \subset G, \alpha_G\}$ is uniquely determined up to orbit equivalence.

Based on these observations, we will show a complete invariant of the inclusion $\mathcal{S} \subset \mathcal{R}$ of type III_0 amenable equivalence relations in terms of the collection of a subgroup H and a normal subgroup G_0 of a finite group G and an ergodic finite group extension of a non-singular flow. This is just a generalization of the Krieger theorem [Kri] by which single equivalence relations $\mathcal{R}$ are classified in terms of a non-singular flow. In type II case, a complete invariant was shown in [Ham2] in terms of a finite group G and a subgroup H which does not contain any non-trivial normal subgroup of G. This is a generalization of the Dye theorem [Dye].

Our main theorem is

THEOREM 6.1: *Up to orbit equivalence, conjugacy of subgroups and conjugacy of extensions of flows, there exists a bijection* Υ*:*

$\{(\mathcal{R}, \mathcal{S}) |$ *$\mathcal{R}$ and $\mathcal{S}$ ($\subset \mathcal{R}$) are ergodic amenable equivalence relations of type* III_0 *and index* $[\mathcal{R}\colon \mathcal{S}] < \infty\}$

$\longrightarrow$

$\{((G, H, G_0), (\{F_t\}, \{S_t\}, \pi)\ ) \mid$
G a finite group, G_0 a normal subgroup of G and $H \subset G$ a subgroup which does not contain any non-trivial normal subgroup of G, and $(\{F_t\}, \{S_t\}, \pi)$ *an ergodic group* $(\mathrm{G}/\mathrm{G}_0)$ extension$\}$.

Here $\{S_t\}$ is an ergodic, non-singular aperiodic flow and $\{F_t\}$ is an ergodic G/G_0-extension of $\{S_t\}$ and π is the factor map, $\pi F_t = S_t \pi$ ($t \in \mathbf{R}$).

In [Ham3], the injectivity of the map Υ is stated in certain restricted cases. The surjectivity is seen in [Ham3] and [Sut2, Theorem 2.3 and Proposition 3.1].

Let us explain the content of each section. In section 2, we recall a canonical system $\{\mathcal{P}, H \subset G, \alpha_G\}$ consisting of a common subrelation $\mathcal{P}$, finite groups $H \subset G$ and an outer action α_G of $\mathcal{P}$ to describe the inclusion $\mathcal{S} \subset \mathcal{R}$ ([Sut1],[Ham2]). Also minimal group covers of finite extensions of transformations are discussed and the uniqueness of the cocycle equipped with a minimal group cover will be crucial in the proof of injectivity of Υ.

In section 3, we show the associated flow $\{F_t^{\mathcal{P}}\}$ of the subrelation $\mathcal{P}$ is an ergodic finite group extension.

In section 4, we show a common discrete decomposition of the equivalence relations $\mathcal{R}, \mathcal{S}$ and $\mathcal{P}$ by using a lacunary measure. Their associated flows are represented by flows built under functions with a common ceiling function determined by the lacunary measure.

In section 5, when the group extension of the flow of data $\Upsilon(\mathcal{R}, \mathcal{S})$ is given in terms of flow built under functions with a common ceiling function, we recover a corresponding lacunary measure, by which the group extension of the flow is defined. This construction will play an important role in proceeding with a copying lemma in the proof of the main theorem.

In section 6 we prove our main theorem. In the proof of injectivity of Υ an idea of a copying lemma is efficiently used by the use of a lacunary measure constructed in section 5, as in [KaWe] where Y. Katznelson and B.Weiss showed a simple proof of the Krieger theorem [Kri].

In section 7, we apply our main theorem for the splitting problem, that is, it is natural to ask when an inclusion of a type III_0 equivalence relation $\mathcal{R}$ and a subrelation $\mathcal{S}$ comes from a type II inclusion by taking respectively a product with a common type III_0 equivalence relation.

In section 8, we investigate essential type III inclusions of subrelations. Quite easily we see that there are lots of non-splitting inclusions. This fact is known by Kosaki and Sano [KoSa] in a different way (using higher relative commutants).

Some of the results in this article were announced in [Ham3].

2. Preliminaries

Let $\mathcal{R}$ be a measured discrete equivalence relation (which we simply call a relation) of a Lebesgue space $(X, \mathcal{B}, m)$. For each $x \in X$ we denote the $\mathcal{R}$-orbit

$$\{y \mid (y, x) \in \mathcal{R}\}$$

of x by $\mathcal{R}(x)$. It is known that there is a non-singular action of a countable group Γ of X such that

$$\mathcal{R}(x) = \{\gamma x | \ \gamma \in \Gamma\} \qquad \text{a.e. } x$$

([FeMo]). When the action by Γ is ergodic, we say $\mathcal{R}$ is ergodic. By $\delta_m(y, x)$ we denote the Radon–Nikodym derivative $\frac{dm\gamma}{dm}(x)$, where $y = \gamma x$ ($\gamma \in \Gamma$). Of course, both the ergodicity and the above description of the Jacobian do not depend on a choice of Γ. $\delta_m(y, x)$ is a cocycle of $\mathcal{R}$. If Γ can be chosen as the group $\mathbf{Z}$, then $\mathcal{R}$ is said to be amenable.

Definition 2.1: By a partial transformation ϕ we mean a collection of measurable subsets A and B and a bijection ϕ: $A \to B$ such that

$$\phi(x) \in \mathcal{R}(x), \qquad \text{a.e. } x \in A.$$

We write Dom $(\phi) = A$, Im $(\phi) = B$ and denote the set of all partial transformations ϕ by $[\mathcal{R}]_*$. In particular, the set of all transforamtions $\phi \in [\mathcal{R}]_*$ such that

$$\text{Dom}(\phi) = \text{Im}(\phi) = X \qquad \text{(up to a null set)}$$

is denoted by $[\mathcal{R}]$ and is called the full group of $\mathcal{R}$. A non-singular transformation ϕ satisfying

$$\phi(\mathcal{R}(x)) = \mathcal{R}(\phi x) \qquad \text{a.e. } x$$

is called a normalizer of $\mathcal{R}$ and the set of all normalizers of $\mathcal{R}$ is denoted by $N[\mathcal{R}]$.

Definition 2.2: Given a measured discrete equivalence relation $\mathcal{S}$ with $\mathcal{S} \subset \mathcal{R}$, we simply call it a subrelation.

Definition 2.3: Let $\mathcal{R}$ (resp. $\mathcal{R}'$) and $\mathcal{S}$ (resp. $\mathcal{S}'$) be a relation and a subrelation on X (X'). If there exists a measure space isomorphism ϕ: $X \to X'$ such that

$$\phi(\mathcal{R}(x)) = \mathcal{R}'(\phi x), \qquad \phi(\mathcal{S}(x)) = \mathcal{S}'(\phi x)$$

we say that the pairs $\mathcal{S} \subset \mathcal{R}$ and $\mathcal{S}' \subset \mathcal{R}'$ are orbit equivalent.

For an ergodic relation $\mathcal{R}$ and a subrelation $\mathcal{S}$, the measurable function $x \in X \mapsto \sharp(\{\mathcal{S}(y) | (y, x) \in \mathcal{R}\})$ is defined and constant $\leq \infty$ a.e. x. By $[\mathcal{R}: \mathcal{S}]$, we denote this constant and call it the index of $\mathcal{S} \subset \mathcal{R}$. Of course, the Jones index ([Jon]) of the Krieger factor and the subfactor constructed from each relation is equal to $[\mathcal{R}: \mathcal{S}]$.

THEOREM 2.1 (Sutherland [Sut1]): *Suppose $\mathcal{R}$ and $\mathcal{S}$ are ergodic and $[\mathcal{R}: \mathcal{S}] < \infty$. Then there exist an ergodic subrelation $\mathcal{P}$ of $\mathcal{S}$, a finite group G and a subgroup H, and an action $g \in G \mapsto \alpha_g \in N[\mathcal{P}]$ satisfying the following conditions (1) and (2):*

(1) *α_G is outer, that is, if $\alpha_g \in [\mathcal{P}]$ then $g = e$.*
(2) *$\mathcal{R} = \mathcal{P} \times_\alpha G$, $\mathcal{S} = \mathcal{P} \times_\alpha H$ where $\mathcal{P} \times_\alpha G$ means an equivalence relation defined by $\mathcal{P} \times_\alpha G(x) = \bigcup_{g \in G} \mathcal{P}(\alpha_g x)$.*

Actually this theorem is more strengthened in the following:

THEOREM 2.2 ([Ham2]):

1. *Under the same assumption as in the previous theorem, we have a collection $\{\mathcal{P}, H \subset G, \alpha_G\}$ satisfying the following condition (3) in addition to (1), (2) in the previous theorem:*
   (3) *H does not contain any non-trivial normal subgroup of G.*
2. *A collection $\{\mathcal{P}, H \subset G, \alpha_G\}$ satisfying (1), (2) and (3) is unique in the sense that if $\mathcal{S} \subset \mathcal{R}$ and $\mathcal{S}' \subset \mathcal{R}'$ are orbit equivalent, then there exists a measure space isomorphism ϕ: $X \to X'$ and a group isomorphism γ: $G \to G'$ with $\gamma(H) = H'$ satisfying*

$$\phi[\mathcal{P}]\phi^{-1} = [\mathcal{P}'],$$
$$\phi \cdot \alpha_g \cdot \phi^{-1} = \alpha'_{\gamma(g)} \qquad (g \in G).$$

Definition 2.4 ([Ham2]): We call the collection $\{\mathcal{P}, H \subset G, \alpha_G\}$ the canonical system for the inclusion $\mathcal{S} \subset \mathcal{R}$.

We recall a couple of notions related to extensions of non-singular transformations. Let S and T be non-singular transformations of Lebesgue spaces $(Y, \mathcal{F}, \mu)$ and $(X, \mathcal{B}, m)$, respectively. We suppose T is a finite extension of S. That is, there is a measurable, non-singular and finite to 1 surjection π: $X \to Y$ satisfying $\pi \cdot T = S \cdot \pi$. ($S$ and π are called a factor of T and a factor map, respectively). Sometimes we call the collection (T, S, π) a (finite to 1) extension. Since the measurable function $y \in Y \to |\pi^{-1}(y)|$ is S-invariant, π is almost everywhere constant to 1 if S is ergodic. In this case, T and X may be assumed to be of the form:

$$X = Y \times \{1, \ldots, k\}, T(y,i) = (Sy, a(y,i)), \pi(y,i) = y, \quad (y,i) \in X$$

where $a(y,i) \in \{1, \ldots, k\}$.

Definition 2.5: Extensions (T, S, π) and (T', S', π') are said to be conjugate if there is a pair of measure space isomorphisms (ϕ, ψ):

$$\phi\colon X \to X', \quad \psi\colon Y \to Y'$$

such that each of them is a conjugacy map:

$$\phi \cdot T = T' \cdot \phi, \quad \psi \cdot S = S' \cdot \psi,$$

and such that

$$\pi' \cdot \phi = \psi \cdot \pi.$$

We call the pair (ϕ, ψ) a conjugacy map of extensions.

Definition 2.6: When T is of the form:

$$T_G(y, g) = (Sy, g \cdot \alpha(y)), \quad (y, g) \in Y \times G$$

where G is a finite group and $\alpha(y) \in G$, T or (T_G, S, π^G) is called a (finite) group extension (or a G-extension) of S. Here $\pi^G(y, g) = y$ is a factor map. If H is a subgroup of G, then we call a transformation

$$T_H(y, [g]_H) = (Sy, [g\alpha(y)]_H), \quad (y, [g]_H) \in Y \times G/H$$

the isometric extension of S determined by H, where $[g]_H$ means a right coset Hg and G/H the right coset space. In this case the map $\pi^H(y, [g]_H) = y$ is a factor map. Of course, the map $\pi^G_H(y, g) = (y, [g]_H)$ is a factor map, too.

An action of G defined by

$$L_g(y, f) = (y, gf), \quad g \in G$$

is called the left translation.

Definition 2.7: If an ergodic extension (T, S, π) admits an ergodic finite group extension T_G of S, the isometric extension (T_H, S, π^H) of S by a subgroup $H \subset G$ and a conjugacy map $\phi\colon Y \times G/H \to X$, $T \cdot \phi = \phi \cdot T_H$, satisfying $\pi \cdot \phi = \pi^H$, then we call the collection (T_G, S, π^G, H) a group cover of (T, S, π).

Definition 2.8: A group cover (T_G, S, π^G, H) of (T, S, π) is said to be minimal if for any group cover $(T_{G'}, S, \pi^{G'}, H')$ of (T, S, π) there exists a non-singular map $\Phi\colon Y \times G' \to Y \times G'$, which is not necessarily invertible, such that $\Phi \cdot T_{G'} = T_G \cdot \Phi$, $\pi^G_H \cdot \Phi = (\phi^{-1} \cdot \phi') \cdot \pi^{G'}_{H'}$ where ϕ (resp. ϕ') is a conjugacy, $T \cdot \phi = \phi \cdot T_H$, $\pi \cdot \phi = \pi^H$ (resp. $T \cdot \phi' = \phi' \cdot T_{H'}$, $\pi \cdot \phi' = \pi^{H'}$).

PROPOSITION 2.1: *Let (T_G, S, π^G) and $(T_{G'}, S', \pi^{G'})$ be finite group extensions. Suppose there are a group isomorphism $\rho'\colon G \to G'$ with $\rho'(H) = H'$ and a conjugacy $\psi\colon Y \to Y'$ of S and S'. Then there exists a measure space isomorphism $\phi\colon Y \times G \to Y' \times G'$ such that (ϕ, ψ) is a conjugacy of these group extensions and such that*

$$\phi \cdot L_g = L_{\rho'(g)} \cdot \phi \qquad (g \in G)$$

if and only if cocycles $\rho'(\alpha)$ and $\alpha'(\psi)$ are cohomologous, that is, there exists a measurable function $v'\colon Y \to G'$ such that

$$\rho'(\alpha(y)) = v'(y) \cdot \alpha'(\psi y) \cdot v'(Sy)^{-1}, \qquad \text{a.e. } y.$$

Proof: ($\Leftarrow$) It suffices to set

$$\phi(y, g) = (\psi y, \rho'(g) \cdot v'(y)).$$

($\Rightarrow$) It suffices to define a measurable function $v'\colon Y \to G'$ by

$$\phi(y, e) = (\psi y, v'(y)). \qquad \blacksquare$$

The following Theorem 2.3 and Theorem 2.4 show us that every ergodic finite extension can be realized by an isometric extension by a subgroup and that the cocycle of the group extension is uniquely determined up to cohomology.

THEOREM 2.3 ([Ham4]): *Let (T, S, π) be an ergodic finite to one extension. Then, there exists a group cover (T_G, S, π^G, H) of (T, S, π) satisfying the condition that H does not contain any non-trivial normal subgroup of G.*

More important is that the cocycle α and the groups H and G are uniquely determined by the condition in Theorem 2.4:

THEOREM 2.4 ([Ham4]): *A group cover (T_G, S, π^G, H) of an ergodic finite extension (T, S, π) is minimal if and only if H does not contain any non-trivial normal subgroup of G. In this case the groups G, H and the cocycle α of T_G are uniquely determined, that is, if we let α' be a cocycle of another minimal cover $(T_{G'}, S, \pi^{G'}, H')$ of (T, s, π) then there exists a group isomorphism $\eta\colon G' \to G$ with $\eta(H') = H$ and a measurable function $v\colon Y \to G$ such that*

$$\eta(\alpha'(y)) = v(y)\alpha(y)v(Sy)^{-1} \qquad \text{a.e. } y.$$

Remark 2.1: If (T, S, π) is a normal extension then the subgroup H of the minimal group cover (T_G, S, π^G, H) of (T, S, π) is trivial ([Ham4]). In this case the theorem is known in Theorem 6.2 of [Zim].

Remark 2.2: As remarked in [Ham4], Theorems 2.3 and 2.4 are also true for flows.

3. Flows associated with a canonical system

Let us briefly recall the construction of the associated flow $F^{\mathcal{R}} = \{F_t^{\mathcal{R}}\}_{t \in \mathbf{R}}$ of a relation $\mathcal{R}$ of a Lebesgue space $(X, \mathcal{B}, m)$.

Definition 3.1: On the product space $X \times \mathbf{R}$ a relation $\tilde{\mathcal{R}}$ is defined by

$$((y, v), (x, u)) \in \tilde{\mathcal{R}} \quad \text{if } (y, x) \in \mathcal{R} \quad \text{and} \quad v = u - \log \delta_m(y, x).$$

By $\zeta_{\tilde{\mathcal{R}}}(x, u)$ we denote the $\tilde{\mathcal{R}}$-ergodic component containing a point $(x, u) \in X \times \mathbf{R}$. By $\mathcal{Z}^{\mathcal{R}}$ we denote the quotient space of $X \times \mathbf{R}$ by the partition of all $\tilde{\mathcal{R}}$-ergodic components. We identify by each $\zeta_{\tilde{\mathcal{R}}}(x, u)$ a point in $\mathcal{Z}^{\mathcal{R}}$ and consider the map $\zeta_{\tilde{\mathcal{R}}}\colon (x, u) \in X \times \mathbf{R} \to \zeta_{\tilde{\mathcal{R}}}(x, u) \in \mathcal{Z}^{\mathcal{R}}$ the natural surjection.

Definition 3.2: The factor flow

$$\zeta_{\tilde{\mathcal{R}}}(x, u) \in \mathcal{Z}^{\mathcal{R}}\colon \to \zeta_{\tilde{\mathcal{R}}}(x, u + t) \in \mathcal{Z}^{\mathcal{R}} \qquad (t \in \mathbf{R})$$

is called the associated flow of $\mathcal{R}$ and denoted by $F^{\mathcal{R}} = \{F_t^{\mathcal{R}}\}_{t \in \mathbf{R}}$.

Let $\mathcal{R}$ and $\mathcal{S}$ be a pair of an ergodic relation and a subrelation of finite index and let $\{\mathcal{P}, H \subset G, \alpha_G\}$ be the canonical system of the incusion $\mathcal{S} \subset \mathcal{R}$.

Definition 3.3: The group G acts on $X \times \mathbf{R}$ by the skew product

$$\tilde{\alpha}_g\colon (x, u) \in X \times \mathbf{R} \to (\alpha_g x, u - \log \delta_m(\alpha_g x, x)) \in X \times \mathbf{R} \qquad (g \in G).$$

Since $\alpha_g \in N[\mathcal{P}]$, a factor transformation $\zeta_{\tilde{\mathcal{P}}}(x, u) \to \zeta_{\tilde{\mathcal{P}}}(\tilde{\alpha}_g(x, u))$ is defined and it commutes with the flow $F_t^{\mathcal{P}}$. By

$$\operatorname{mod}_{\tilde{\mathcal{P}}} \alpha_g$$

we denote the factor transformation acting on $\mathcal{Z}^{\mathcal{P}}$. We define

$$G_0 = \{g \in G | \quad \operatorname{mod}_{\tilde{\mathcal{P}}} \alpha_g = \mathrm{Id}\}$$

which is a normal subgroup of G. We also denote the quotient group G/G_0 by K and the subgroup $\{[h]_{G_0} |\ h \in H\}$ by L, where $[g]_{G_0}$ is the coset $G_0 \cdot g$. We set, for each $q \in K$,

$$A_q = \operatorname{mod}_{\tilde{\mathcal{P}}} \alpha_g$$

where $g \in G$ with $[g]_{G_0} = q$. Obviously, A_K is a free action of the group K which commutes with the flow $F_t^{\mathcal{P}}$.

We have three flows $F^{\mathcal{P}}$, $F^{\mathcal{S}}$ and $F^{\mathcal{R}}$ associated with the canonical system $\{\mathcal{P}, H \subset G.\alpha_G\}$. If we observe the inclusions

$$\zeta_{\tilde{\mathcal{P}}}(x,u) \subset \zeta_{\tilde{\mathcal{S}}}(x,u) \subset \zeta_{\tilde{\mathcal{R}}}(x,u),$$

then maps $\pi_{\mathcal{R}}^{\mathcal{P}}$: $\mathcal{Z}^{\mathcal{P}} \to \mathcal{Z}^{\mathcal{R}}$, $\pi_{\mathcal{S}}^{\mathcal{P}}$: $\mathcal{Z}^{\mathcal{P}} \to \mathcal{Z}^{\mathcal{S}}$ and $\pi_{\mathcal{R}}^{\mathcal{S}}$: $\mathcal{Z}^{\mathcal{S}} \to \mathcal{Z}^{\mathcal{R}}$ are naturally defined by

$$\begin{aligned}
\pi_{\mathcal{R}}^{\mathcal{P}} &: \zeta_{\tilde{\mathcal{P}}}(x,u) \to \zeta_{\tilde{\mathcal{R}}}(x,u), \\
\pi_{\mathcal{S}}^{\mathcal{P}} &: \zeta_{\tilde{\mathcal{P}}}(x,u) \to \zeta_{\tilde{\mathcal{S}}}(x,u), \\
\pi_{\mathcal{R}}^{\mathcal{S}} &: \zeta_{\tilde{\mathcal{S}}}(x,u) \to \zeta_{\tilde{\mathcal{R}}}(x,u).
\end{aligned}$$

As a matter of fact, this observation will work well for getting a complete invariant for orbit equivalence of inclusions.

THEOREM 3.1: *The flow $\mathcal{F}^{\mathcal{P}}$ is a K-extension of the flow $\mathcal{F}^{\mathcal{R}}$, and $(\mathcal{F}^{\mathcal{P}}, \mathcal{F}^{\mathcal{R}}, \pi_{\mathcal{R}}^{\mathcal{P}}, L)$ is a group cover of the extension $(\mathcal{F}^{\mathcal{S}}, \mathcal{F}^{\mathcal{R}}, \pi_{\mathcal{R}}^{\mathcal{S}})$.*

Proof: Since A_K freely acts on the flow space $\mathcal{Z}^{\mathcal{P}}$, we get a cross section $E \subset \mathcal{Z}^{\mathcal{P}}$ of positive measure for this action. Let $\zeta_{\tilde{\mathcal{P}}}(x,u) \in \mathcal{Z}^{\mathcal{P}}$, then $q \in K$ is uniquely determined by

$$A_q^{-1}\zeta_{\tilde{\mathcal{P}}}(x,u) \in E.$$

In other words, the singleton $\{A_q^{-1}\zeta_{\tilde{\mathcal{P}}}(x,u)\}$ is the intersection of the A_K-orbit of the point $\zeta_{\tilde{\mathcal{P}}}(x,u)$ and the set E. Therefore we can define measure space isomorphisms $\psi_{\mathcal{P}}$: $\mathcal{Z}^{\mathcal{P}} \to E \times K$, $\psi_{\mathcal{S}}$: $\mathcal{Z}^{\mathcal{S}} \to E \times K/L$ and $\psi_{\mathcal{R}}$: $\mathcal{Z}^{\mathcal{R}} \to E$ by setting

$$\begin{aligned}
\psi_{\mathcal{P}} &: \zeta_{\tilde{\mathcal{P}}}(x,u) \to (A_q^{-1}\zeta_{\tilde{\mathcal{P}}}(x,u), q) \in E \times K, \\
\psi_{\mathcal{S}} &: \zeta_{\tilde{\mathcal{S}}}(x,u) \to (A_q^{-1}\zeta_{\tilde{\mathcal{P}}}(x,u), [q]_L) \in E \times K/L, \\
\psi_{\mathcal{R}} &: \zeta_{\tilde{\mathcal{R}}}(x,u) \to A_q^{-1}\zeta_{\tilde{\mathcal{P}}}(x,u) \in E.
\end{aligned}$$

A remarkable and simple fact is that the action $q \in K \to \psi_{\mathcal{P}} \cdot A_q \cdot \psi_{\mathcal{P}}^{-1}$ is the left translation L_q, $q \in K$, of $E \times K$. Let us check it.

$$\psi_{\mathcal{P}} \cdot A_{q'} \cdot \psi_{\mathcal{P}}^{-1}(A_q^{-1}\zeta_{\tilde{\mathcal{P}}}(x,u), q) = \psi_{\mathcal{P}} \cdot A_{q'}\zeta_{\tilde{\mathcal{P}}}(x,u).$$

Since $A_{q'q}^{-1}(A_{q'}\zeta_{\tilde{\mathcal{P}}}(x,u)) = A_q^{-1}(\zeta_{\tilde{\mathcal{P}}}(x,u)) \in E$,

$$\psi_{\mathcal{P}} \cdot A_{q'}\zeta_{\tilde{\mathcal{P}}}(x,u) = (A_q^{-1}(\zeta_{\tilde{\mathcal{P}}}(x,u)), q'q) = L_{q'}(A_q^{-1}\zeta_{\tilde{\mathcal{P}}}(x,u), q).$$

Next, we are going to show that $F^{\mathcal{P}}$ is a K-extension of $F^{\mathcal{R}}$. By π^K we denote the projection map $(z,q) \in E \times K \to z$. We see that $F^{\mathcal{R}}$ is the factor flow of $F^{\mathcal{P}}$ acting on the quotient space of $\mathcal{Z}^{\mathcal{P}}$ taken by the decomposition by A_K-orbits. This means

$$\pi^K \cdot \psi_{\mathcal{P}} \cdot F_t^{\mathcal{P}} = \psi_{\mathcal{R}} \cdot F_t^{\mathcal{R}} \cdot \pi_{\mathcal{R}}^{\mathcal{P}}.$$

Therefore $\pi^K \cdot \psi_{\mathcal{P}} \cdot F_t^{\mathcal{P}} \cdot \psi_{\mathcal{P}}^{-1}(z,e) = \pi^K \cdot \psi_{\mathcal{P}} \cdot F_t^{\mathcal{P}}\zeta_{\tilde{\mathcal{P}}}(x,u) = \psi_{\mathcal{R}}F_t^{\mathcal{R}}\zeta_{\tilde{\mathcal{R}}}(x,u) = \psi_{\mathcal{R}}F_t^{\mathcal{R}}\psi_{\mathcal{R}}^{-1}z$, where $\psi_{\mathcal{P}}\zeta_{\tilde{\mathcal{P}}}(x,u) = (z,e)$. Hence, we may write

$$\psi_{\mathcal{P}}F_t^{\mathcal{P}}\psi_{\mathcal{P}}^{-1}(z,e) = (\psi_{\mathcal{R}}F_t^{\mathcal{R}}\psi_{\mathcal{R}}^{-1}z, k(t,z)),$$

where $k(t,z) \in K$. For any $q \in K$,

$$\begin{aligned}
\psi_{\mathcal{P}}F_t^{\mathcal{P}}\psi_{\mathcal{P}}^{-1}(z,q) &= L_q\psi_{\mathcal{P}}F_t^{\mathcal{P}}\psi_{\mathcal{P}}^{-1}(z,e) \\
&= \psi_{\mathcal{P}}A_qF_t^{\mathcal{P}}\psi_{\mathcal{P}}^{-1}(z,e) \\
&= \psi_{\mathcal{P}}F_t^{\mathcal{P}}A_q\psi_{\mathcal{P}}^{-1}(z,e) \\
&= \psi_{\mathcal{P}}F_t^{\mathcal{P}}\psi_{\mathcal{P}}^{-1}L_q(z,e) \\
&= L_q(\psi_{\mathcal{R}}F_t^{\mathcal{R}}\psi_{\mathcal{R}}^{-1}z, k(t,z)) \\
&= (\psi_{\mathcal{R}}F_t^{\mathcal{R}}\psi_{\mathcal{R}}^{-1}z, q \cdot k(t,z)).
\end{aligned}$$

Thus the flow $F_t^{\mathcal{P}}$ is a K-extension of the flow $F_t^{\mathcal{R}}$. Finally let us check that the flow $F^{\mathcal{S}}$ is an isometric extension determined by the subgroup L. To see this, we note that

$$\psi_{\mathcal{S}}^{-1} \cdot \pi_L^K = \pi_{\mathcal{S}}^{\mathcal{P}} \cdot \psi_{\mathcal{P}}^{-1}.$$

In fact,

$$\pi_{\mathcal{S}}^{\mathcal{P}} \cdot \psi_{\mathcal{P}}^{-1}(A_q^{-1}\zeta_{\tilde{\mathcal{P}}}(x,u), q) = \pi_{\mathcal{S}}^{\mathcal{P}}\zeta_{\tilde{\mathcal{P}}}(x,u) = \zeta_{\tilde{\mathcal{S}}}(x,u)$$

and

$$\psi_{\mathcal{S}}^{-1} \cdot \pi_L^K(A_q^{-1}\zeta_{\tilde{\mathcal{P}}}(x,u), q) = \psi_{\mathcal{S}}^{-1}(A_q^{-1}\zeta_{\tilde{\mathcal{P}}}(x,u), [q]_L) = \zeta_{\tilde{\mathcal{S}}}(x,u).$$

Therefore,

$$\begin{aligned}\psi_{\mathcal{S}} F_t^{\mathcal{S}} \psi_{\mathcal{S}}^{-1}(z,[q]_L) &= \psi_{\mathcal{S}} F_t^{\mathcal{S}} \psi_{\mathcal{S}}^{-1} \pi_L^K(z,q) \\ &= \psi_{\mathcal{S}} F_t^{\mathcal{S}} \pi_{\mathcal{S}}^{\mathcal{P}} \psi_{\mathcal{P}}^{-1}(z,q) \\ &= \psi_{\mathcal{S}} \pi_{\mathcal{S}}^{\mathcal{P}} F_t^{\mathcal{P}} \psi_{\mathcal{P}}^{-1}(z,q) \\ &= \pi_L^K \psi_{\mathcal{P}} F_t^{\mathcal{P}} \psi_{\mathcal{P}}^{-1}(z,q) \\ &= (\psi_{\mathcal{R}} F_t^{\mathcal{R}} \psi_{\mathcal{R}}^{-1} z, [q \cdot k(t,z)]_L). \qquad \blacksquare\end{aligned}$$

PROPOSITION 3.1: *Suppose $\mathcal{S} \subset \mathcal{R}$ and $\mathcal{S}' \subset \mathcal{R}'$ are orbit equivalent. Then the corresponding subgroups $H \subset G$ and $H' \subset G'$ are conjugate and the extensions $(F^{\mathcal{P}}, F^{\mathcal{R}}, \pi_{\mathcal{R}}^{\mathcal{P}})$ and $(F^{\mathcal{P}'}, F^{\mathcal{R}'}, \pi_{\mathcal{R}'}^{\mathcal{P}'})$ are conjugate.*

Proof: It follows from Theorem (2.2) that there exist a measure space isomorphism ϕ: $X \to X'$ and a group isomorphism ρ: $G \to G'$ with $\rho(H) = H'$ such that

1. $\phi[\mathcal{P}]\phi^{-1} = [\mathcal{P}']$,
2. $\phi \cdot \alpha_g \cdot \phi^{-1} = {\alpha'}_{\rho(g)}$ $(g \in G)$.

Define a measure space isomorphism $\tilde{\phi}$: $X \times \mathbf{R} \to X' \times \mathbf{R}$ by setting

$$\tilde{\phi}(x,u) = \left(\phi x, u - \log \frac{dm'\phi}{dm}(x)\right).$$

The map $\tilde{\phi}$ induces a measure space isomorphism $\mathcal{Z}^{\mathcal{P}} \to \mathcal{Z}^{\mathcal{P}'}$ by restricting $\tilde{\phi}$ to the quotient space $\mathcal{Z}^{\mathcal{P}}$ in such a way that

$$\zeta_{\tilde{\mathcal{P}}}(x,u) \to \zeta_{\tilde{\mathcal{P}}'}(\tilde{\phi}(x,u)).$$

By Ψ we denote this restriction. We also restrict $\tilde{\phi}$ to the quotient space $\mathcal{Z}^{\mathcal{R}}$ in such a way that

$$\bigcup_{g\in G} \zeta_{\tilde{\mathcal{P}}}(\tilde{\alpha_g}(x,u)) \to \bigcup_{g\in G} \zeta_{\tilde{\mathcal{P}}'}(\tilde{\phi} \cdot \tilde{\alpha_g}(x,u)).$$

By ψ we denote this restriction $\mathcal{Z}^{\mathcal{R}} \to \mathcal{Z}^{\mathcal{R}'}$. Then it is not difficult to see that $\psi \cdot \pi_{\mathcal{R}}^{\mathcal{P}} = \pi_{\mathcal{R}'}^{\mathcal{P}'} \cdot \Psi$, $\Psi \cdot F_t^{\mathcal{P}} = F_t^{\mathcal{P}'} \cdot \Psi$ and $\psi \cdot F_t^{\mathcal{R}} = F_t^{\mathcal{R}'} \cdot \psi$. $\blacksquare$

4. Common discrete decomposition

From the assumption that $\mathcal{R}$ is of type III_0, it is well known that $\mathcal{R}$ admits an equivalent σ-finite lacunary measure μ so that almost every ergodic measure appearing on each ergodic component of the ergodic decomposition of the measure $d\mu \times e^u du$ is non-atomic (and σ-finite, infinite). As α_G is a finite group action, we may assume that μ is an α_G-invariant measure. Othewise, we replace μ by the average of the measures $\mu \cdot \alpha_g$, $g \in G$. The measure obtained in this way is also a lacunary measure for $\mathcal{P}$.

In the case of a single relation, so-called discrete decomposition of the relation ([HaOs]) is known. As for a pair of a relation and a subrelation, we will have a corresponding common discrete decomposition. We also represent flows $F^{\mathcal{P}}$, $F^{\mathcal{S}}$ and $F^{\mathcal{R}}$ and in terms of flows built under functions with a common ceiling function. This will help us show our main theorem. Set

$$\mathcal{P}_\mu = \mathrm{Ker}(\delta_\mu) = \{(z,x) \in \mathcal{P} | \delta_\mu(z,x) = 1\},$$

then $\mathcal{P}_\mu$ is a subrelation. Likewise, $\mathcal{R}_\mu$ and $\mathcal{S}_\mu$ are defined. The discrete decomposition of $\mathcal{P}$ tells us that $\mathcal{P}$ admits an $R \in N[\mathcal{P}_\mu] \cap [\mathcal{P}]$ and $c > 0$ satisfying

$$\log \delta_\mu(Rx,x) = \min\{\log \delta_\mu(y,x) | y \in X, (y,x) \in \mathcal{P}, \delta_\mu(y,x) > 1\} > c,$$

and

$$\mathcal{P} = \mathcal{P}_\mu \times_R \mathbf{Z}.$$

We set

$$f_\mu^{\mathcal{P}}(x) = \log \delta_\mu(Rx,x).$$

We note that $f_\mu^{\mathcal{P}}(x)$ is a $\mathcal{P}_\mu$-invariant function.

LEMMA 4.1: *The measurable function $f_\mu^{\mathcal{P}}(x)$ is α_G-invariant.*

Proof: Let $(y,x) \in \mathcal{R}$. Then, y is of the form $y = \alpha_g z$ for some $g \in G$ and z with $(z,x) \in \mathcal{P}$. Since

$$\delta_\mu(y,x) = \delta_\mu(y,z)\delta_\mu(z,x) = \delta_\mu(z,x)$$

we see that for a.e. x,

$$f_\mu^{\mathcal{P}}(x) = \min\{\log \delta_\mu(y,x) | (y,x) \in \mathcal{R}, \delta_\mu(y,x) > 1\}.$$

Hence, for any $g \in G$,

$$\begin{aligned} f_\mu^{\mathcal{P}}(\alpha_g x) &= \log \delta_\mu(R\alpha_g x, \alpha_g x) \\ &= \log \delta_\mu(R\alpha_g x, x) \\ &\geq \min\{\log \delta_\mu(y,x) | (y,x) \in \mathcal{R}, \delta_\mu(y,x) > 1\} \\ &= f_\mu^{\mathcal{P}}(x). \end{aligned}$$

As G is a group, the inequality is actually the equality. ∎

LEMMA 4.2: $\alpha_G \subset N[\mathcal{P}_\mu]$.

Proof: This follows from $\alpha_G \subset N[\mathcal{P}]$ and that μ is α_G-invariant. ∎

Definition 4.1: For each $x \in X$, we denote by $\zeta_{\mathcal{P}_\mu}(x)$ the $\mathcal{P}_\mu$-ergodic component containing x. $W^{\mathcal{P}_\mu}$ is the quotient space of X by the partition consisting of all $\mathcal{P}_\mu$-ergodic components. As usual each $\zeta_{\mathcal{P}_\mu}(x)$ is identified with a point in $W^{\mathcal{P}_\mu}$. Likewise, $W^{\mathcal{S}_\mu}$ and $W^{\mathcal{R}_\mu}$ are defined.

By Lemma 4.1 and Lemma 4.2, we see that $f_\mu^{\mathcal{P}}$ is a function of $W^{\mathcal{R}_\mu}$, that is, there exists a measurable function f of $W^{\mathcal{R}_\mu}$ such that

$$f_\mu^{\mathcal{P}}(x) = f(\zeta_{\mathcal{R}_\mu}(x)).$$

LEMMA 4.3:

$$\alpha_g \cdot R \cdot \alpha_g^{-1} \in R \cdot [\mathcal{P}_\mu] \quad (g \in G).$$

Proof: Since $\alpha_g Rx \in \mathcal{P}(\alpha_g x)$ and $R\alpha_g x \in \mathcal{P}(\alpha_g x)$, $(R\alpha_g x, \alpha_g Rx) \in \mathcal{P}$. We have

$$\begin{aligned} \delta_\mu(R\alpha_g x, \alpha_g Rx) &= \delta_\mu(R\alpha_g x, \alpha_g x)\delta_\mu(\alpha_g x, x)\delta_\mu(x, Rx)\delta_\mu(Rx, \alpha_g Rx) \\ &= \delta_\mu(Rx, x) \cdot 1 \cdot \delta_\mu(x, Rx) \cdot 1 = 1. \end{aligned}$$

That is, $(R\alpha_g x, \alpha_g Rx) \in \mathcal{P}_\mu$. ∎

We are ready to present a common discrete decomposition for both $\mathcal{R}$ and $\mathcal{S}$.

PROPOSITION 4.1:

$$R \in N[\mathcal{P}_\mu] \cap N[\mathcal{S}_\mu] \cap N[\mathcal{R}_\mu],$$
$$\mathcal{R}_\mu = \mathcal{P}_\mu \times_\alpha G, \quad \mathcal{R} = (\mathcal{P}_\mu \times_\alpha G) \times_R \mathbf{Z}$$

and

$$\mathcal{S}_\mu = \mathcal{P}_\mu \times_\alpha H, \quad \mathcal{S} = (\mathcal{P}_\mu \times_\alpha H) \times_R \mathbf{Z}.$$

Proof: It immediately follows from $R^{-1}[\mathcal{P}_\mu]R = [\mathcal{P}_\mu]$ and $R^{-1}\alpha_g R \in [\mathcal{P}_\mu]\alpha_g$ that $R \in N[\mathcal{P}_\mu \times_\alpha G] \cap N[\mathcal{P}_\mu \times_\alpha H]$. ∎

Definition 4.2: Let $\pi^{\mathcal{P}_\mu}_{\mathcal{S}_\mu}: \zeta_{\mathcal{P}_\mu}(x) \in W^{\mathcal{P}_\mu} \to \zeta_{\mathcal{S}_\mu}(x) \in W^{\mathcal{S}_\mu}$ be the the natural surjection. Likewise $\pi^{\mathcal{S}_\mu}_{\mathcal{R}_\mu}$ is defined. The transformation R induces factor tranformations $U^{\mathcal{P}_\mu}, U^{\mathcal{S}_\mu}$ and $U^{\mathcal{R}_\mu}$ of $W^{\mathcal{P}_\mu}, W^{\mathcal{S}_\mu}$ and $W^{\mathcal{R}_\mu}$ respectively as follows:

$$U^{\mathcal{P}_\mu}(\zeta_{\mathcal{P}_\mu}(x)) = \zeta_{\mathcal{P}_\mu}(Rx),$$
$$U^{\mathcal{S}_\mu}(\zeta_{\mathcal{S}_\mu}(x)) = \zeta_{\mathcal{S}_\mu}(Rx),$$
$$U^{\mathcal{R}_\mu}(\zeta_{\mathcal{R}_\mu}(x)) = \zeta_{\mathcal{R}_\mu}(Rx).$$

Definition 4.3: Due to Lemma 4.3, the action α_G induces a group action on $W^{\mathcal{P}_\mu}$ as its factor:

$$\mathrm{mod}_{\mathcal{P}_\mu} \alpha_g: \zeta_{\mathcal{P}_\mu}(x) \in W^{\mathcal{P}_\mu} \to \zeta_{\mathcal{P}_\mu}(\alpha_g x) \in W^{\mathcal{P}_\mu}, \qquad g \in G.$$

It is easy to see that

$$G_0 = \{g \in G \mid \mathrm{mod}_{\mathcal{P}_\mu} \alpha_g = \mathrm{Id}\};$$

we obtain a free action β_K of the group K of $W^{\mathcal{P}_\mu}$ defined by

$$\beta_{[g]_{G_0}} = \mathrm{mod}_{\mathcal{P}_\mu} \alpha_g, \qquad [g]_{G_0} \in K.$$

By Lemma 4.3, each β_r, $r \in K$, commutes with $U^{\mathcal{P}_\mu}$.

We need a terminology on flow built under a function.

Definition 4.4: Given a measure space W and a non-singular transformation U and a measurable function f defined on W and of positive values we set

$$(W, f) = \{(w, u) \mid w \in W, \ \ 0 \le u < f(w)\},$$

and denote by $\{(W, U, f)_t\}_{t \in \mathbf{R}}$ the flow built under functions with a ceiling function f and a base transformation U. We denote by $f(n, w)$ the cocyle determined by f and U, that is, $f(1, w) = f(w)$, $f(2, w) = f(w) + f(Uw)$ and so on.

The following proposition describes Theorem 3.1 in terms of flow built under the function.

PROPOSITION 4.2: *Let μ be an equivalent σ-finite lacunary measure of $\mathcal{R}$ which is α_G invariant.*

1. *There exist an ergodic non-singular transformation U of a measure space E and a group extension U_K: $(\omega, q) \in E \times K \to (U\omega, q \cdot k(\omega)) \in E \times K$ and a conjugacy map $(\psi_{\mathcal{P}}, \psi_{\mathcal{R}})$ of the extensions $(U^{\mathcal{P}_\mu}, U^{\mathcal{R}_\mu}, \pi^{\mathcal{P}_\mu}_{\mathcal{R}_\mu})$ and (U_K, U, π^K) such that*

$$\psi_{\mathcal{P}} \cdot \beta_q = L_q \cdot \psi_{\mathcal{P}} \qquad (q \in K).$$

2. *The flow spaces $\mathcal{Z}^{\mathcal{P}}$ and $\mathcal{Z}^{\mathcal{R}}$ are naturally identified with the spaces $(W^{\mathcal{P}_\mu}, f \cdot \pi^{\mathcal{P}_\mu}_{\mathcal{R}_\mu})$ and $(W^{\mathcal{R}_\mu}, f)$ respectively, on which each flow built under a function acts. Through this identification,*

$$(\psi_{\mathcal{P}} \times \mathrm{Id}) \cdot F^{\mathcal{P}}_t \cdot (\psi_{\mathcal{P}} \times \mathrm{Id})^{-1} = (E \times K, U_K, f(\omega) \times \mathrm{Id})_t,$$
$$(\psi_{\mathcal{R}} \times \mathrm{Id}) \cdot F^{\mathcal{R}}_t \cdot (\psi_{\mathcal{R}} \times \mathrm{Id})^{-1} = (E, U, f(\omega))_t,$$
$$(\psi_{\mathcal{P}} \times \mathrm{Id}) \cdot A_q \cdot (\psi_{\mathcal{P}} \times \mathrm{Id})^{-1} = L_q \times \mathrm{Id}, \quad q \in K$$

where A_q is identified with $\beta_q \times \mathrm{Id}$.

Proof: (1) Consider a partition $\{\bigcup_{g \in G} \zeta_{\mathcal{P}_\mu}(\alpha_g x) | \ x \in X\}$ of X. Since G is finite, this is a measurable partition. Actually, each set $\bigcup_{g \in G} \zeta_{\mathcal{P}_\mu}(\alpha_g x) = \cup_{q \in K} \beta_q \zeta_{\mathcal{P}_\mu}(x)$ coincides with the $\mathcal{P}_\mu \times_\alpha G$-ergodic component $\zeta_{\mathcal{R}_\mu}(x)$. Since β_K is a finite group action, one can get a cross section $E \subset W^{\mathcal{P}_\mu}$ of positive measure, i.e. each β_K-orbit only once intersects with E. β_K is free. So, there exists for each $\zeta_{\mathcal{P}_\mu}(x)$ a unique $q \in K$ such that $\beta_{q^{-1}} \zeta_{\mathcal{P}_\mu}(x) \in E$. Hence, $W^{\mathcal{P}_\mu}$ is identified with $E \times K$ by the measure isomorphism:

$$\psi_{\mathcal{P}}: \zeta_{\mathcal{P}_\mu}(x) \in W^{\mathcal{P}_\mu} \to (\beta_{q^{-1}} \zeta_{\mathcal{P}_\mu}(x), q) \in E \times K.$$

Then, for each $q' \in K$

$$\begin{aligned} \beta_{q'} \psi_{\mathcal{P}}^{-1}(\beta_{q^{-1}} \zeta_{\mathcal{P}_\mu}(x), q) &= \beta_{q'} \zeta_{\mathcal{P}_\mu}(x) \\ &= \psi_{\mathcal{P}}^{-1} \psi_{\mathcal{P}} \zeta_{\mathcal{P}_\mu}(\alpha_{g'} x) \quad \text{(where } [g']_{G_0} = q') \\ &= \psi_{\mathcal{P}}^{-1}(\beta_{q^{-1}} \zeta_{\mathcal{P}_\mu}(x), q'q) \\ &= \psi_{\mathcal{P}}^{-1} L_{q'}(\beta_{q^{-1}} \zeta_{\mathcal{P}_\mu}(x), q), \end{aligned}$$

that is, β_K is conjugate with the left translation L_K.

Define a non-singular transformation U acting on E and a measurable function $\omega \in E \to k(\omega) \in K$ by setting

$$(U\omega, k(\omega)) = \psi_{\mathcal{P}} \cdot U^{\mathcal{P}_\mu} \cdot \psi_{\mathcal{P}}^{-1}(\omega, e), \qquad \omega \in E,$$

where e is the unit of K. Let U_K be a group extension

$$U_K(\omega, q) = (U\omega, q \cdot k(\omega)), ((\omega, q) \in E \times K).$$

Then for each $q \in K$,

$$\begin{aligned}\psi_{\mathcal{P}} \cdot U^{\mathcal{P}_\mu} \cdot \psi_{\mathcal{P}}^{-1}(\omega, q) &= \psi_{\mathcal{P}} \cdot U^{\mathcal{P}_\mu} \cdot \beta_q \cdot \psi_{\mathcal{P}}^{-1}(\omega, e)\\ &= \psi_{\mathcal{P}} \cdot \beta_q \cdot U^{\mathcal{P}_\mu} \cdot \psi_{\mathcal{P}}^{-1}(\omega, e)\\ &= \psi_{\mathcal{P}} \cdot \beta_q \cdot \psi_{\mathcal{P}}^{-1}(U\omega, k(\omega))\\ &= (U\omega, q \cdot k(\omega)) = U_K(\omega, q).\end{aligned}$$

Therefore, $\psi_{\mathcal{P}}$ is a conjugacy map of $U^{\mathcal{P}_\mu}$ and U_K. By $\psi_{\mathcal{R}}$, we denote a measure space isomorphism:

$$\psi_{\mathcal{R}}\colon \zeta_{\mathcal{R}_\mu}(y) \in W^{\mathcal{R}_\mu} \to \omega \in E$$

where $\omega \in E$ and $q \in K$ such that $\zeta_{\mathcal{R}_\mu}(y) = \dot{\bigcup}_{q \in K} \beta_q \omega$. Then we easily see that these satisfy

$$\psi_{\mathcal{R}} \cdot \pi^{\mathcal{P}_\mu}_{\mathcal{R}_\mu} = \pi^K \cdot \psi_{\mathcal{P}}.$$

Let us check that $\psi_{\mathcal{R}}$ is a conjugacy map. Let $\zeta_{\mathcal{R}_\mu}(y) \in W^{\mathcal{R}_\mu}$, $\omega \in E$ and $\zeta_{\mathcal{R}_\mu}(y) = \cup_{q \in K} \beta_q \omega$. Then,

$$\begin{aligned}\zeta_{\mathcal{R}_\mu}(Ry) &= U^{\mathcal{R}_\mu} \zeta_{\mathcal{R}_\mu}(y) & &= U^{\mathcal{R}_\mu}\Big(\bigcup_{q \in K} \beta_q \omega\Big)\\ &= U^{\mathcal{P}_\mu}\Big(\bigcup_{q \in K} \beta_q \omega\Big) & &= \bigcup_{q \in K} U^{\mathcal{P}_\mu} \beta_q \omega\\ &= \bigcup_{q \in K} \psi_{\mathcal{P}}^{-1} \cdot \psi_{\mathcal{P}} U^{\mathcal{P}_\mu} \psi_{\mathcal{P}}^{-1} \cdot \psi_{\mathcal{P}}(\beta_q \omega) & &= \bigcup_{q \in K} \psi_{\mathcal{P}}^{-1} U_K(\omega, q)\\ &= \bigcup_{q \in K} \psi_{\mathcal{P}}^{-1}(U\omega, q \cdot k(\omega)) & &= \bigcup_{q \in K} \beta_{q \cdot k(\omega)} U\omega\\ &= \bigcup_{q \in K} \beta_q U\omega.\end{aligned}$$

Thus,

$$\psi_{\mathcal{R}} U^{\mathcal{R}_\mu}(\zeta_{\mathcal{R}_\mu}(y)) = \psi_{\mathcal{R}}(\zeta_{\mathcal{R}_\mu}(Ry)) = U\omega = U\psi_{\mathcal{R}}(\zeta_{\mathcal{R}_\mu}(y)).$$

(2) immediately follows from (1). ∎

5. Lacunary measure

Let $\mathcal{R}$ and $\mathcal{S}$ be an ergodic relation and a subrelation of a measure space $(X, \mathcal{B}, m)$ with index $[\mathcal{R}: \mathcal{S}] < \infty$. Let $\{\mathcal{P}, H \subset G, \alpha_G\}$ be the canonical system for the inclusion $\mathcal{S} \subset \mathcal{R}$. We may assume m is α_G-invariant and otherwise replace m by the average of the measures $m \cdot \alpha_g$, $g \in G$. Let G_0, K, L be defined as in section 3. Suppose that there exist a K-extension $U_K(z, q) = (Uz, q \cdot k(z))$, $((z, q) \in E \times K)$, a measurable function $f(z)$ on E with $\inf f(z) > 0$ and that the extension $(F^{\mathcal{P}}, F^{\mathcal{R}}, \pi^{\mathcal{P}}_{\mathcal{R}})$ is conjugate with an extension between the corresponding flows built under a function. Then a natural question, how we recover an equivalent σ-finite lacunary and α_G-invariant measure μ so that U_K, U_L, U and f coincide with ones in Proposition 4.2, arises. In case of a single relation $\mathcal{R}$, this is known (e.g. [HaOs]). For simplicity in this section let us denote the flows built under functions $(E \times K, U_K, f(z) \times \mathrm{Id})$ and $(E, U, f(z))$ by $\{B^K_t\}$ and $\{B_t\}$.

THEOREM 5.1: *As above we suppose that the extension $(F^{\mathcal{P}}, F^{\mathcal{R}}, \pi^{\mathcal{P}}_{\mathcal{R}})$ is conjugate with the extension $(B^K, B, \pi^K \times \mathrm{Id})$. Then there exist an equivalent σ-finite lacunary and α_G-invariant measure μ and measure space isomorphisms $\psi_{\mathcal{R}}: W^{\mathcal{R}_\mu} \to E$, $\psi_{\mathcal{P}}: W^{\mathcal{P}_\mu} \to E \times K$ satisfying the following properties:*

1. *$(\psi_{\mathcal{P}}, \psi_{\mathcal{R}})$ is a conjugacy of extensions $(U^{\mathcal{P}_\mu}, U^{\mathcal{R}_\mu}, \pi^{\mathcal{P}_\mu}_{\mathcal{R}_\mu})$ and (U_K, U, π_K).*
2. *$\psi_{\mathcal{P}} \cdot \beta_q = L_q \cdot \psi_{\mathcal{P}}$.*
3. *If $R \in N[\mathcal{P}_\mu] \cap [\mathcal{P}]$ is the corresponding normalizer admitted in the discrete decomposition of $\mathcal{P}$ with respect to μ, then*

$$\log \frac{d\mu R}{d\mu}(x) = f(\psi_{\mathcal{R}}(\zeta_{\mathcal{R}_\mu}(x))).$$

After a couple of lemmas, the proof will be completed. First of all, due to Proposition 2.1 and Theorem 2.4, a conjugacy of group extensions implies that there exists a conjugacy map $(\pi_{\mathcal{P}}, \pi_{\mathcal{R}})$ of both the extensions, where $\pi_{\mathcal{P}}: W^{\mathcal{P}_\mu} \to E \times K$ and $\pi_{\mathcal{R}}: W^{\mathcal{R}_\mu} \to E$, satisfying

$$\pi_{\mathcal{P}} \cdot A_q \cdot \pi_{\mathcal{P}}^{-1} = L_q \times \mathrm{Id} \qquad (q \in K).$$

We define measurable functions $x \in X \to \zeta(x) \in E$, $x \in X \to q(x) \in K$ and $x \in X \to \eta(x)$: $0 \le \eta(x) < f(\zeta(x))$ by setting

$$\pi_{\mathcal{P}} \zeta_{\tilde{\mathcal{P}}}(x, 0) = (\zeta(x), q(x), \eta(x)).$$

LEMMA 5.1: $\zeta(\alpha_g x) = \zeta(x)$, $\eta(\alpha_g x) = \eta(x)$, $q(\alpha_g x) = [g]_{G_0} \cdot q(x)$.

Proof:

$$\begin{aligned}(\zeta(\alpha_g x), q(\alpha_g x), \eta(\alpha_g x)) &= \pi_{\mathcal{P}} \zeta_{\tilde{\mathcal{P}}}(\alpha_g x, 0) \\ &= \pi_{\mathcal{P}} \cdot A_{[g]_{G_0}} \zeta_{\tilde{\mathcal{P}}}(x, 0) \\ &= (L_{[g]_{G_0}} \times \mathrm{Id}) \cdot \pi_{\mathcal{P}} \zeta_{\tilde{\mathcal{P}}}(x, 0) \\ &= (\zeta(x), [g]_{G_0} \cdot q(x), \eta(x)). \qquad \blacksquare\end{aligned}$$

LEMMA 5.2: *There exists a uniquely determined cocycle* n: $(x, y) \in \mathcal{P} \to n(x, y) \in \mathbf{Z}$ *such that*

$$\eta(x) = \eta(y) + \log \delta_m(x, y) - f(n(x, y), \zeta(y)).$$

Hence

$$U^{n(x,y)} \zeta(y) = \zeta(x), \qquad \text{a.e. } (x, y) \in \mathcal{P}.$$

Proof: We note that for $(x, y) \in \mathcal{P}$, $B_{\log \delta_m(x,y)} \cdot \pi_{\mathcal{P}}(y, 0) = \pi_{\mathcal{P}}(x, 0)$. So,

$$\begin{aligned}(\zeta(x), q(x), \eta(x)) =& B_{\log \delta_m(x,y)}(\zeta(y), q(y), \eta(y)) \\ =&(U^n \zeta(y), q(y) \cdot k(n, \zeta(y)), \eta(y) + \log \delta_m(x, y) - f(n, \zeta(y))\end{aligned}$$

where $k(n, z) \in K$ is the cocycle defined by $k(z)$ and U, that is, $k(1, z) = k(z)$, $k(2, z) = k(z) \cdot k(Uz)$ etc. and $n = n(x, y)$ is such that

$$f(n, \zeta(y)) \le \eta(y) + \log \delta_m(x, y) < f(n + 1, \zeta(y)).$$

In this case,

$$\eta(x) = \eta(y) + \log \delta_m(x, y) - f(n, \zeta(y)). \qquad \blacksquare$$

Now we are going to define equivalence relations on $X \times \mathbf{Z}$. An equivalence relation $\overline{\mathcal{P}}$ is defined by

$$((x, l), (y, l - n(x, y)) \in \overline{\mathcal{P}} \quad \text{if } (x, y) \in \mathcal{P}.$$

Let $\overline{R}$ and $\overline{\alpha}_g$, $g \in G$ be the transformations of $X \times \mathbf{Z}$ defined by

$$\overline{R}(x, l) = (x, l + 1), \quad (x, l) \in X \times \mathbf{Z}$$

and

$$\overline{\alpha}_g(x, l) = (\alpha_g x, l).$$

We see that $\overline{R} \in N[\overline{\mathcal{P}}]$. Therefore, the relation $\overline{\mathcal{P}} \times_{\overline{R}} \mathbf{Z}$ is defined and denoted by $\mathcal{G}(\mathcal{P})$. Clearly, $\overline{\alpha}_G \subset N[\mathcal{G}(\mathcal{P})]$. So, relations $\mathcal{G}(\mathcal{S}) = \mathcal{G}(\mathcal{P}) \times_{\overline{\alpha}} H$ and $\mathcal{G}(\mathcal{R}) = \mathcal{G}(\mathcal{P}) \times_{\overline{\alpha}} G$ are defined.

LEMMA 5.3: *For a.e. x, there exists y such that $(y,x) \in \mathcal{P}$ and $n(y,x) = 1$.*

Proof: Set

$$E_0 = \{(x,u) \in X \times \mathbf{R} | \ -\eta(x) \leq u < -\eta(x) + f(\zeta(x))\}$$

and

$$E_1 = \{(x,u) \in X \times \mathbf{R} | \ -\eta(x) + f(\zeta(x)) \leq u < -\eta(x) + f(2,\zeta(x))\}.$$

Then the smallest $\tilde{\mathcal{P}}$-invariant set containing E_0 is the whole space $X \times \mathbf{R}$. In particular, for almost every point in E_1, its $\tilde{\mathcal{P}}$-orbit intersects with E_0. Therefore, for a.e. $x \in X$, there exists a $t \in \mathbf{R}$ and a $y \in X$ such that $(x,y) \in \mathcal{P}$, $(x,t) \in E_1$, $(y, t - \log\delta_m(y,x)) \in E_0$. Hence, $((x, f(\zeta(x)) - \eta(x) + \eta(y)), (y,0)) \in \tilde{\mathcal{P}}$ and $\eta(y) \leq \log\delta_m(y,x) < \eta(y) + f(\eta(x))$. Hence $n(y,x) = 1$. ∎

Proof of Theorem 5.1: On the measure space $X \times \mathbf{Z}$ we define an inclusion $\mathcal{G}(\mathcal{S}) \subset \mathcal{G}(\mathcal{R})$, whose canonical system is $\{\mathcal{G}(\mathcal{P}), H \subset G, \overline{\alpha}_G\}$. We will show that the pair of a relation and a subrelation admits a lacunary measure $\overline{\mu}$ corresponding to the measure in Proposition 4.2 and that the inclusion $\mathcal{G}(\mathcal{S}) \subset \mathcal{G}(\mathcal{R})$ is orbit equivalent with $\mathcal{S} \subset \mathcal{R}$.

Consider a σ-finite measure $\overline{\mu}$ on $X \times \mathbf{Z}$ defined by

$$d\overline{\mu}(x,i) = dm(x)e^{-\eta(x)+f(i,\zeta(x))}.$$

The measure $\overline{\mu}$ is $\overline{\mathcal{P}} \times_{\overline{\alpha}} G$-invariant and $\mathcal{G}(\mathcal{P})$-lacunary. In fact, for $(x,y) \in \mathcal{P}$

$$\begin{aligned}
d\overline{\mu}(y, i - n(y,x)) &= dm(y)e^{-\eta(y)+f(i+n(y,x),\zeta(y))} \\
&= dm(y)e^{\phi(y,x)-\eta(y)+f(i+n(y,x),\zeta(y))} \\
&= dm(y)e^{-\eta(x)-f(n(x,y),\zeta(y))+f(i+n(x,y),\zeta(y))} \\
&= dm(y)e^{-\eta(x)+f(i,\zeta(x))} \\
&= d\overline{\mu}(x,i), \\
d\overline{\mu}(\overline{\alpha_g}(x,i)) &= dm(\alpha_g x)e^{-\eta(\alpha_g x)+f(i,\zeta(\alpha_g x))} \\
&= dm(x)e^{-\eta(x)+f(i,\zeta(x))} \\
&= d\overline{\mu}(x,i),
\end{aligned}$$

and

$$\frac{d\overline{\mu}\overline{R}}{d\overline{\mu}}(x,l) = e^{f(U^l\zeta(x))},$$

where the right hand admits a positive lower bound > 1. We note $\mathcal{G}(\mathcal{P})_{\overline{\mu}} = \overline{\mathcal{P}}$. Therefore, $\mathcal{G}(\mathcal{P}) = \overline{\mathcal{P}} \times_{\overline{R}} \mathbf{Z} = \mathcal{G}(\mathcal{P})_{\overline{\mu}} \times_{\overline{R}} \mathbf{Z}$.

Next, we consider a measurable partition of $X \times \mathbf{Z}$ consisting of all $\mathcal{G}(\mathcal{P})_{\overline{\mu}}$ ergodic components. As usual, we denote for each $(x,i) \in X \times \mathbf{Z}$, the $\mathcal{G}(\mathcal{P})_{\overline{\mu}}$ ergodic component containing (x,i) by $\zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(x,i)$. On $W^{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}$, $W^{\mathcal{G}(\mathcal{S})_{\overline{\mu}}}$ and $W^{\mathcal{G}(\mathcal{R})_{\overline{\mu}}}$ respectively we define non-singular transformations $\overline{U}_{\mathcal{P}}$, $\overline{U}_{\mathcal{S}}$ and $\overline{U}_{\mathcal{R}}$ by setting

$$\overline{U}_{\mathcal{P}}(\zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(x,i)) = \zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(x,i+1),$$
$$\overline{U}_{\mathcal{S}}(\zeta_{\mathcal{G}(\mathcal{S})_{\overline{\mu}}}(x,i)) = \zeta_{\mathcal{G}(\mathcal{S})_{\overline{\mu}}}(x,i+1),$$

and

$$\overline{U}_{\mathcal{R}}(\zeta_{\mathcal{G}(\mathcal{R})_{\overline{\mu}}}(x,i)) = \zeta_{\mathcal{G}(\mathcal{R})_{\overline{\mu}}}(x,i+1).$$

The action $\overline{\alpha}_G$ induces a free action $\overline{\beta}_K$ on $W^{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}$ by

$$\overline{\beta}_{[g]_{G_0}}\zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(x,i) = \zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(\alpha_g x, i).$$

Consider maps $\theta_{\mathcal{P}}\colon W^{\mathcal{G}(\mathcal{P})_{\overline{\mu}}} \to E \times K$ and $\theta_{\mathcal{R}}\colon W^{\mathcal{G}(\mathcal{R})_{\overline{\mu}}} \to E$ defined respectively by

$$\theta_{\mathcal{P}}(\zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(x,0)) = (\zeta(x), q(x)), \quad x \in X$$

and

$$\theta_{\mathcal{R}}(\zeta_{\mathcal{G}(\mathcal{R})_{\overline{\mu}}}(x,0)) = \zeta(x), \quad x \in X.$$

We know that for a.e. $(x,i) \in X \times \mathbf{Z}$ there exists a point $y \in X$ such that $((x,i),(y,0)) \in \overline{\mathcal{P}}$. Therefore maps $\theta_{\mathcal{P}}$ and $\theta_{\mathcal{R}}$ are well defined measure space isomorphisms and satisfy

$$\begin{aligned}\theta_{\mathcal{P}}\overline{U}_{\mathcal{P}}(\zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(x,0)) &= \theta_{\mathcal{P}}(\zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(x,1))\\ &= \theta_{\mathcal{P}}(\zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(y,0))\\ &= (\zeta(y), q(y))\\ &= U_K(\zeta(x), q(x))\\ &= U_K\theta_{\mathcal{P}}\zeta_{\mathcal{G}}(\mathcal{P})_{\overline{\mu}}(x,0),\end{aligned}$$

where $(y,x)\in\mathcal{P}$ and $n(y,x)=-1$. Moreover,

$$\begin{aligned}\theta_{\mathcal{P}}\overline{\beta}_{[g]_{G_0}}(\zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(x,0)) &= \theta_{\mathcal{P}}\zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(\alpha_g x,0)\\ &= (\zeta(\alpha_g x), q(\alpha_g x))\\ &= (\zeta(x), \gamma([g]_{G_0})q(x))\\ &= L_{[g]_{G_0}}(\zeta(x),q(x))\\ &= L_{[g]_{G_0}}\theta_{\mathcal{P}}\zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(x,0).\end{aligned}$$

Therefore,

$$\begin{aligned}\theta_{\mathcal{R}}\overline{U}_{\mathcal{R}}(\zeta_{\mathcal{G}(\mathcal{R})_{\overline{\mu}}}(x,0)) &= \theta_{\mathcal{P}}\overline{U}_{\mathcal{P}}\{\overline{\beta}_q(\zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(x,0))|\ q\in K\}\\ &= U_K\{\theta_{\mathcal{P}}\overline{\beta}_q(\zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(x,0))|\ q\in K\}\\ &= U_K\{L_{\gamma(q)}\theta_{\mathcal{P}}(\zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(x,0))|\ q\in K\}\\ &= U_K\{L_{\gamma(q)}(\zeta(x),q(x))|\ q\in K\}\\ &= U_K(\{\zeta(x)\}\times K)\\ &= U\zeta(x)\\ &= U\theta_{\mathcal{R}}(\zeta_{\mathcal{G}(\mathcal{R})_{\overline{\mu}}}(x,0)).\end{aligned}$$

Next consider the restriction of the inclusion $\mathcal{G}(\mathcal{S})\subset\mathcal{G}(\mathcal{R})$ to the set $X\times\{0\}$. Since $\mathcal{G}(\mathcal{S})$ is of type III, there exists a $u\in[\mathcal{G}(\mathcal{S})]_*$ such that $\operatorname{Dom}u=X\times\mathbf{Z}$, $\operatorname{Im}u=X\times\{0\}$. If we identify $X\times\{0\}$ with X, then the restriction $\mathcal{G}(\mathcal{R})|_{X\times\{0\}}\supset\mathcal{G}(\mathcal{S})|_{X\times\{0\}}$ is nothing but $\mathcal{R}\supset\mathcal{S}$. Therefore, $\mathcal{G}(\mathcal{R})\supset\mathcal{G}(\mathcal{S})$ is orbit equivalent with $\mathcal{R}\supset\mathcal{S}$. These pairs of a relation and a subrelation admit the canonical systems $(\mathcal{P},H\subset G,\alpha_G)$ and $(\mathcal{G}(\mathcal{P}),H\subset G,\overline{\alpha}_G)$. Thanks to the uniqueness of a canonical system (Theorem 2.2), we have a measure space isomorphism $v\colon X\to X\times\mathbf{Z}$ and a group isomorphism ρ of G preserving the subgroup H such that

$$\begin{aligned}v\cdot[\mathcal{P}]\cdot v^{-1} &= [\mathcal{G}(\mathcal{P})],\\ v\cdot\alpha_{\rho(g)}\cdot v^{-1} &= \overline{\alpha}_g,\quad \forall g\in G.\end{aligned}$$

Finally, we define a measure μ and a normalizer R by setting

$$\begin{aligned}\mu(\cdot) &= \overline{\mu}(v(\cdot)),\\ R &= v^{-1}\cdot\overline{R}\cdot v.\end{aligned}$$

Then $v^{-1}\cdot[\overline{\mathcal{P}}]\cdot v=[\mathcal{G}(\mathcal{P})_{\overline{\mu}}]\cdot v=[\mathcal{P}_\mu]$, and hence

$$\mathcal{P}=v^{-1}\cdot\overline{\mathcal{P}}\times_{\overline{R}}\mathbf{Z}\cdot v=v^{-1}\cdot\overline{\mathcal{P}}\cdot v\times_R\mathbf{Z}=\mathcal{P}_\mu\times_R\mathbf{Z}.$$

The isomorphism v: $X \to X \times \mathbf{Z}$ induces a measure space isomophism $V_{\mathcal{P}}$: $W^{\mathcal{P}_\mu} \to W^{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}$ by restricting v on $W^{\mathcal{P}_\mu}$, that is,

$$V_{\mathcal{P}}(\zeta_{\mathcal{P}_\mu}(x)) = \zeta_{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}(v(x)).$$

Likewise, a measure space isomorphism $V_{\mathcal{R}}$: $W^{\mathcal{R}_\mu} \to W^{\mathcal{G}(\mathcal{R})_{\overline{\mu}}}$:

$$V_{\mathcal{R}}(\zeta_{\mathcal{R}_\mu}(x)) = \zeta_{\mathcal{G}(\mathcal{R})_{\overline{\mu}}}(v(x))$$

is defined. Then,

$$\begin{aligned}\overline{U}_{\mathcal{P}_\mu} \cdot V_{\mathcal{P}}(\zeta_{\mathcal{P}_\mu}(x)) &= \overline{U}_{\mathcal{P}_\mu}(\zeta_{\mathcal{P}_\mu}(v(x)) = \zeta_{\mathcal{P}_\mu}(\overline{R} \cdot v(x)) \\ &= \zeta_{\mathcal{P}_\mu}(v \cdot R(x)) = V_{\mathcal{P}} \cdot U^{\mathcal{P}_\mu}(\zeta_{\mathcal{P}_\mu}(x)).\end{aligned}$$

Similarly, we have

$$\overline{U}_{\mathcal{R}_\mu} \cdot V_{\mathcal{R}} = V_{\mathcal{R}} \cdot U^{\mathcal{R}_\mu}.$$

Obviously, by definition of the maps $V_{\mathcal{P}}$ and $V_{\mathcal{R}}$

$$\pi^{\mathcal{G}(\mathcal{P})_{\overline{\mu}}}_{\mathcal{G}(\mathcal{R})_{\overline{\mu}}} \cdot V_{\mathcal{P}} = V_{\mathcal{R}} \cdot \pi^{\mathcal{P}_\mu}_{\mathcal{R}_\mu}.$$

Finally, let us define measure space isomorphisms:

$$\begin{aligned}\psi_{\mathcal{P}} &= \theta_{\mathcal{P}} \cdot V_{\mathcal{P}}, \\ \psi_{\mathcal{R}} &= \theta_{\mathcal{R}} \cdot V_{\mathcal{R}}.\end{aligned}$$

Then we see that

$$\psi_{\mathcal{P}} \cdot \beta_q = L_q \cdot \psi_{\mathcal{P}}.$$

Let $x \in X$ and set $v(x) = (y, l)$. By Lemma 5.3, $z \in X$ is obtained so that

$$(y, z) \in \mathcal{P}, \quad \text{and } n(y, z) = l.$$

We note

$$\begin{aligned}\theta_{\mathcal{R}} V_{\mathcal{R}}(\zeta_{\mathcal{R}_\mu}(x)) &= \theta_{\mathcal{R}}(\zeta_{\mathcal{G}(\mathcal{R})_{\overline{\mu}}}(v(x))) \\ &= \theta_{\mathcal{R}}(\zeta_{\mathcal{G}(\mathcal{R})_{\overline{\mu}}}(y, l)) \\ &= \theta_{\mathcal{R}}(\zeta_{\mathcal{G}(\mathcal{R})_{\overline{\mu}}}(z, 0)) \\ &= \zeta(z).\end{aligned}$$

Therefore,

$$\begin{aligned}\log\frac{d\mu R}{d\mu}(x) &= \log\frac{d\overline{\mu}R}{d\overline{\mu}}(y,l)\\ &= \log\frac{d\overline{\mu}R}{d\overline{\mu}}(z,0))\\ &= f(\zeta(z))\\ &= f(\theta_{\mathcal{R}}V_{\mathcal{R}}(\zeta_{\mathcal{R}_\mu}(x)))\\ &= f(\psi_{\mathcal{R}}(\zeta_{\mathcal{R}_\mu}(x))).\end{aligned}$$

Thus the measure μ and these conjugacy maps $\theta_{\mathcal{P}}$ and $\theta_{\mathcal{R}}$ satisfy the properties in the theorem. ∎

6. Complete invariant

In this section we will show a complete invariant for orbit equivalence of inclusions $\mathcal{S} \subset \mathcal{R}$ of amenable type III_0 ergodic measured discrete equivalence relations with finite index.

We set

$$\mathcal{D} = \{((G,H,G_0), \quad (\{F_t\},\{S_t\},\pi))\}$$

where:

1. G is a finite group, G_0 a normal subgroup of G and H a subgroup of G which does not contain any non-trivial normal subgroup of G,
2. $\{S_t\}$ is an ergodic, aperiodic non-singular flow, $\{F_t\}$ is an ergodic G/G_0-extension of $\{S_t\}$ and π is the factor map, $\pi \cdot F_t = S_t \cdot \pi$ $(t \in \mathbf{R})$.

Let us define a map Υ:

$$\begin{aligned}\{(\mathcal{R},\mathcal{S}) \quad | \quad & \mathcal{R} \text{ an amenable ergodic relation of type } \mathrm{III}_0 \text{ and}\\ & \mathcal{S} \text{ an ergodic subrelation with } [\mathcal{R}:\mathcal{S}] < \infty\}\\ \longrightarrow \quad & \mathcal{D}\end{aligned}$$

by setting

$$\Upsilon(\mathcal{R},\mathcal{S}) = ((G,H,G_0), \ (\{F_t^{\mathcal{P}}\},\{F_t^{\mathcal{R}}\},\pi_{\mathcal{R}}^{\mathcal{P}})),$$

where $\{\mathcal{P}, H \subset G, \alpha_G\}$ is the canonical system for $\mathcal{S} \subset \mathcal{R}$ and $G_0 = \{g \in G|\ \ \mathrm{mod}_{\tilde{\mathcal{P}}}\,\alpha_g = \mathrm{Id}\}$.

Definition 6.1: We say that

$$((G,H,G_0),(\{F_t\},\{S_t\},\pi)) \text{ and } ((G',H',G'_0),(\{F'_t\},\{S'_t\},\pi')) \text{ in } \mathcal{D}$$

are conjugate if there is a group isomorphism ρ: $G \to G'$ with $\rho(H) = H'$, $\rho(G_0) = G'_0$ and if the group extensions $(\{F_t\},\{S_t\},\pi))$ and $(\{F'_t\}, \{S'_t\}, \pi'))$ are conjugate.

THEOREM 6.1: *The map Υ is bijective up to orbit equivalence and conjugacy.*

Proof: We already saw in Proposition 3.1 that if $\mathcal{S} \subset \mathcal{R}$ and $\mathcal{S}' \subset \mathcal{R}'$ are orbit equivalent then $\Upsilon(\mathcal{R},\mathcal{S})$ and $\Upsilon(\mathcal{R}',\mathcal{S}')$ are conjugate. In order to see the surjectivity of Υ let us construct a model having the given data $((G,H,G_0),(\{F_t\},\{S_t\},\pi)) \in \mathcal{D}$. Let $K = G/G_0$ and

$$F_t(z,k) = (S_t z, k\cdot\theta(t,z)), (z,k) \in Y \times K$$

where $\theta(t,z) \in K$, and

$$\pi(z,k) = z.$$

Let $Y = \prod_{n=-\infty}^{\infty} G$ and σ be the Bernoulli shift on Y with the infinite product measure of the uniform measure on each coordinate space G. Let ϕ be a type III_1 non-singular transformation of a Lebesgue space (Ω, P). Let $\Gamma \subset \mathbf{R}$ be an arbitrary countable dense subgroup. Construct a product measure space

$$X = \Omega \times Z \times K \times \mathbf{R} \times Y \times G.$$

Let us define the transformations $\overline{S}_\gamma$ $(\gamma \in \Gamma)$, $\overline{\phi}, \overline{\sigma}$ and α_l $(l \in G)$ acting on X as follows. For $(\omega,z,k,u,y,q) \in X$, set

$$\begin{aligned}
\overline{S}_\gamma(\omega,z,k,u,y,g) =& (\omega, S_\gamma z, k\cdot\theta(\gamma,z), u+\gamma-\log\delta_\mu(S_\gamma z,z), y, g),\\
\overline{\phi}(\omega,z,k,u,y,g) =& (\phi\omega, z, k, u-\log\delta_P(\phi\omega,\omega), y, g),\\
\overline{\sigma}(\omega,z,k,u,y,g) =& (\omega,z,k,u,\sigma y, y_0\cdot g),\\
\alpha_l(\omega,z,k,u,y,g) =& (\omega, z, [l]_{G_0}\cdot k, u, y, g\cdot l^{-1}).
\end{aligned}$$

Here y_0 denotes the 0-th coordinate of y. Let $\mathcal{R}$ (resp. $\mathcal{S}$) be the relation generated by the transformations $\overline{S}_\gamma$, $\gamma \in \Gamma$, $\overline{\phi}, \overline{\sigma}$ and α_l, $l \in G$ (resp. $\overline{S}_\gamma$, $\gamma \in \Gamma$, $\overline{\phi}, \overline{\sigma}$ and α_l, $l \in H$). The action α_G commutes with both $\overline{S}_\Gamma$ and $\overline{\sigma}$. So if we let $\mathcal{P}$ be the relation generated by $\overline{S}_\gamma$,$\gamma \in \Gamma$, $\overline{\phi}$, and $\overline{\sigma}$, then $\{\mathcal{P}, H \subset G, \alpha_G\}$ gives us the canonical system for the inclusion $\mathcal{S} \subset \mathcal{R}$. Moreover, it is not hard to see that

1. $\{g \in G | \mathrm{mod}_{\tilde{\mathcal{P}}} \alpha_g = \mathrm{id.}\} = G_0$,

2. $\mathcal{Z}^{\mathcal{P}} = Z \times K$, $\mathcal{Z}^{\mathcal{R}} = Z$, $\pi_{\mathcal{R}}^{\mathcal{P}}(z,k) = \pi(z,k) = z$,

3. $F_t^{\mathcal{P}} = F_t$, $F_t^{\mathcal{R}} = S_t$.

Next, we will show the injectivity of Υ (up to orbit equivalence and conjugacy) via Proposition 6.1. The key of the proof is to replace a given conjugacy map of factor maps $\pi_{\mathcal{R}}^{\mathcal{P}}$ and $\pi_{\mathcal{R}'}^{\mathcal{P}'}$ by a conjugacy map which commutes with the left translations and to apply the copying lemma developed by Y. Katznelson and B. Weiss [KaWe].

Assume

$$\Upsilon(\mathcal{R}, \mathcal{S}) = \big((G, H, G_0),\ (\{F_t^{\mathcal{P}}\}, \{F_t^{\mathcal{R}}\}, \pi_{\mathcal{R}}^{\mathcal{P}})\big)$$

and

$$\Upsilon(\mathcal{R}', \mathcal{S}') = \Big((G', H', G_0'),\ (\{F_t^{\mathcal{P}'}\}, \{F_t^{\mathcal{R}'}\}, \pi_{\mathcal{R}'}^{\mathcal{P}'})\Big)$$

are conjugate. We may and do assume $G = G'$, $H = H'$, $G_0 = G_0'$. We set $K = G/G_0$ and $L = \{[h]_{G_0} |\ h \in H\}$. Choose and fix an equivalent σ-finite infinite lacunary measure μ which is α_G-invariant. We let $R \in N[\mathcal{P}_\mu] \cap [\mathcal{P}]$ be the normalizer of the discrete decomposition of $\mathcal{P}$ with respect to μ and let $f(\cdot)$ be the measurable function of $W^{\mathcal{R}_\mu}$ such that

$$\log \frac{d\mu R}{d\mu}(x) = f(\zeta_{\mathcal{R}_\mu}(x)).$$

Corresponding to the measure μ, Proposition 4.2 says that there are measure space isomorphisms $\psi_{\mathcal{P}}$: $W^{\mathcal{P}_\mu} \to E \times K$, $\psi_{\mathcal{R}}$: $W^{\mathcal{R}_\mu} \to E$ and an ergodic non-singular transformation U of the measure space E and a group extension U_K: $(\omega, q) \in E \times K \to (U\omega, q \cdot k(\omega)) \in E \times K$ satisfying the following properties:

1. $(\psi_{\mathcal{P}}, \psi_{\mathcal{R}})$ is the conjugacy map of the extensions $(U^{\mathcal{P}_\mu}, U^{\mathcal{R}_\mu}, \pi_{\mathcal{R}_\mu}^{\mathcal{P}_\mu})$ and (U_K, U, π^K).

2. $\psi_{\mathcal{P}} \cdot \beta_q = L_q \cdot \psi_{\mathcal{P}}$ $(q \in K)$.

The flow spaces $\mathcal{Z}^{\mathcal{P}}$ and $\mathcal{Z}^{\mathcal{R}}$ are naturally identified with the spaces $(W^{\mathcal{P}_\mu}, f \cdot \pi_{\mathcal{R}_\mu}^{\mathcal{P}_\mu})$ and $(W^{\mathcal{R}_\mu}, f)$ respectively, on which each flow built under a function acts. Through this identification,

1. $$\begin{aligned}(\psi_{\mathcal{P}} \times \mathrm{Id}) \cdot F_t^{\mathcal{P}} \cdot (\psi_{\mathcal{P}} \times \mathrm{Id})^{-1} &= (E \times K, U_K, f(\omega) \times \mathrm{Id})_t,\\ (\psi_{\mathcal{R}} \times \mathrm{Id}) \cdot F_t^{\mathcal{R}} \cdot (\psi_{\mathcal{R}} \times \mathrm{Id})^{-1} &= (E, U, f(\omega))_t,\end{aligned}$$

2. $$(\psi_{\mathcal{P}} \times \mathrm{Id}) \cdot A_q \cdot (\psi_{\mathcal{P}} \times \mathrm{Id})^{-1} = L_q \times \mathrm{Id}, \quad q \in K,$$

where A_q is identified with $\beta_q \times \mathrm{Id}$.

Our assumption means that there exist a measure space E', a non-singular transforamtion U', a K-extension U'_K and measure space isomorphisms ϕ: $E \to E'$ and Φ: $E \times K \to E' \times K$ such that

$$\Phi \cdot U_K = U'_K \cdot \Phi, \qquad \phi \cdot U = U' \cdot \phi$$

and such that the extensions $(F^{\mathcal{P}'}, F^{\mathcal{R}'}, \pi^{\mathcal{P}'}_{\mathcal{R}'})$ and $(U'_K, U', f' \times \mathrm{Id})$ are conjugate, where $f'(\cdot) = f(\phi^{-1}(\cdot))$. Then due to Theorem 2.4, we can choose the above map Φ so that

$$\begin{aligned}\Phi \cdot U_K =& U'_K \cdot \Phi,\\ \Phi \cdot L_q =& L_q \cdot \Phi \qquad (q \in K).\end{aligned}$$

This observation will be very important in the proof of injectivity. (See also the remark after the proof of Theorem 6.1.)

If we apply Theorem 5.1, we obtain an equivalent σ-finite lacunary measure μ' of $\mathcal{R}'$ which is α'_G-invariant, a non-singular transformation $R' \in N[\mathcal{P}'_{\mu'}] \cap [\mathcal{P}']$ and measure space isomorphisms $\psi_{\mathcal{P}'}$: $W^{\mathcal{P}'_{\mu'}} \to E' \times K$ and $\psi_{\mathcal{R}'}$: $W^{\mathcal{R}'_{\mu'}} \to E'$ satisfying the following properties:

1. $(\psi_{\mathcal{P}'}, \psi_{\mathcal{R}'})$ is a conjugacy of the group extensions $(U^{\mathcal{P}'_{\mu'}}, U^{\mathcal{R}'_{\mu'}}, \pi^{\mathcal{P}'_{\mu'}}_{\mathcal{R}'_{\mu'}})$ and (U'_K, U', π'^K),
2. $f^{\mathcal{R}'_{\mu'}}(\cdot) = f'(\psi_{\mathcal{R}'}(\cdot))$,
3. $\psi_{\mathcal{P}'} \cdot \beta'_q = L_q \cdot \psi_{\mathcal{P}'}$.

Thus, we have shown the following proposition:

PROPOSITION 6.1: *If $\Upsilon(\mathcal{R}, \mathcal{S})$ and $\Upsilon(\mathcal{R}', \mathcal{S}')$ are conjugate, then there exist equivalent σ-finite lacunary α_G (resp. α'_G)-invariant measures μ and μ', non-singular transformations $R \in N[\mathcal{P}_\mu]$ and $R' \in N[\mathcal{P}'_{\mu'}]$, measurable functions f on $W^{\mathcal{R}}_\mu$ and f' on $W^{\mathcal{R}'}_{\mu'}$ of positive lower bound respectively, and a measure space isomorphism θ: $W^{\mathcal{P}}_\mu \to W^{\mathcal{P}'}_{\mu'}$ satisfying the following properties:*

1. $\theta \cdot U^{\mathcal{P}_\mu} = U^{\mathcal{P}'_{\mu'}} \cdot \theta, \qquad \theta \cdot \mathrm{mod}_{\mathcal{P}_\mu} \alpha_g = \mathrm{mod}_{\mathcal{P}'_{\mu'}} \alpha'_g \cdot \theta \qquad (g \in G),$
2. $\mathcal{P} = \mathcal{P}_\mu \times_R \mathbf{Z}, \quad \mathcal{P}' = \mathcal{P}'_{\mu'} \times_{R'} \mathbf{Z},$

3. $\log \frac{d\mu R}{d\mu}(x) = f(\zeta_{\mathcal{R}_\mu}(x))$, $\log \frac{d\mu' R'}{d\mu'}(x') = f'(\zeta_{\mathcal{R}'_{\mu'}}(x'))$, *and* $f(\zeta_{\mathcal{R}_\mu}(x)) = f'(\theta(\zeta_{\mathcal{R}_\mu}(x)))$, *where* $\beta_q = \mathrm{mod}_{\mathcal{P}_\mu}\,\alpha_g$, *and* $\beta'_q = \mathrm{mod}_{\mathcal{P}_{\mu'}}\,\alpha'_g$ $(q = [g]_{G_0})$.

The above proposition will show us how well an idea of the copying lemma developed by Y.Katznelson and B.Weiss ([KaWe]) works in a proof of our main theorem.

Definition 6.2:

1. A tower $\xi = (\mathcal{P}_\xi, \mathcal{T}_\xi)$ on a measurable subset $E \subset X$ consists of a finite partition $\mathcal{P}_\xi = \{E_i | i \in \Lambda\}$ of E, and a finite family of partial transformations $\mathcal{T}_\xi = \{e_{i,j} \in [\mathcal{R}]_* |\ i, j \in \Lambda\}$ satisfying
$$\mathrm{Dom}(e_{i,j}) = E_j, \qquad \mathrm{Im}(e_{i,j}) = E_i,$$
$$e_{i,j} \cdot e_{j,k} = e_{i,k}, \qquad e_{i,i} = \mathrm{Id}|_{E_i}.$$
We call each subset E_i a cell of the tower. The tower ξ is also considered as the finite subrelation on E defined by $\{(e_{i,j}x, x) |\ x \in E_j,\ i, j \in \Lambda\}$.

2. Let $\xi_i, i = 1, 2$ be towers on a measurable subset E, and let $\mathcal{P}_{\xi_i} = \{E_\alpha | \alpha \in \Lambda_i\}$ and $\mathcal{T}_{\xi_i} = \{e_{\alpha,\beta} |\ \alpha, \beta \in \Lambda_i\}$. We say that ξ_2 refines ξ_1 if
$$\Lambda_2 = \Lambda_1 \times \Gamma \qquad (\Gamma \text{ a finite set }),$$
$$E_\alpha = \bigcup_{\gamma \in \Gamma} E_{(\alpha,\gamma)} \qquad (\alpha \in \Lambda_1) \quad \text{and}$$
$$e_{(\alpha,\gamma),(\beta,\gamma)} = e_{\alpha,\beta} \ \text{ on } \ E_{(\beta,\gamma)} \qquad (\alpha, \beta \in \Lambda_1, \gamma \in \Gamma).$$
Choose and fix an $\alpha \in \Lambda_1$, and define the tower $\eta = (\mathcal{P}_\eta, \mathcal{T}_\eta)$ on E_α by setting
$$\mathcal{P}_\eta = \{E_{(\alpha,\gamma)} |\ \gamma \in \Gamma\}, \quad \mathcal{T}_\eta = \{e_{(\alpha,\gamma),(\alpha,\gamma')} |\ \gamma, \gamma' \in \Gamma\};$$
then we denote ξ_2 by $\xi_1 \times \eta$ and call it a product tower.

From now on we write
$$U = U^{\mathcal{P}_\mu}, \quad U' = U^{\mathcal{P}'_{\mu'}}, \quad [g] = [g]_{G_0}.$$

By $f(z, w)$, where $w \in W^{\mathcal{P}_\mu}$ and $z = U^n w$ for some n, we mean the cocycle $f(n, w)$.

As usual, we identify each ergodic component $\zeta_{\mathcal{P}_\mu}(x)$ with a point in $W^{\mathcal{P}_\mu}$ and consider a map $x \in X \to \zeta_{\mathcal{P}_\mu}(x) \in W^{\mathcal{P}_\mu}$ as the natural surjection. Let us take a probability measure ν on $W^{\mathcal{P}_\mu}$ which is β_K-invariant and is equivalent with the pushforward measure $\mu(\zeta_{\mathcal{P}_\mu}^{-1}(\cdot))$. Then there exist uniquely determined conditional measures $\mu(\cdot|\omega)$ supported on the fibre $\{x \in X | \zeta_{\mathcal{P}_\mu}(x) = \omega\}$, $\omega \in W^{\mathcal{P}_\mu}$ such that for any $f \in L^1(X, \mu)$,

$$\int f(x) d\mu(x) = \int_{W^{\mathcal{P}_\mu}} \nu(d\omega) \int_{\zeta_{\mathcal{P}_\mu}(x)=\omega} f(x)\mu(dx|\omega).$$

We note that almost all measures $\mu(\cdot|\omega)$ are non-atomic and σ-finite and infinite. Likewise, by setting

$$\nu'(\cdot) = \nu(\theta^{-1}(\cdot))$$

we have the conditional measures $\mu'(\cdot|\omega')$, $\omega' \in W^{\mathcal{P}'_{\mu'}}$, i.e.

$$\int f'(x) d\mu'(x') = \int_{W^{\mathcal{P}'_{\mu'}}} \nu'(d\omega') \int_{\zeta_{\mathcal{P}'_{\mu'}}(x')=\omega'} f'(x')\mu'(dx'|\omega').$$

We note that ν' is β'_K invariant.

Continuation of the proof of Theorem 6.1: We are finally ready to complete the proof of Theorem 6.1. Let us note that every amenable ergodic relation admits a refining sequence of towers of the whole space satisfying that the countable union of the corresponding increasing finite subrelations coincides with the whole relation and that all cells of the towers generate the whole σ-algebra.

Firstly we choose and fix measurable subsets $X_0 \subset X$ and $X'_0 \subset X'$ with

$$\mu(X_0|\omega) = \mu'(X'_0|\theta(\omega)) < \infty, \qquad \text{a.e. } \omega \in W^{\mathcal{R}_\mu}.$$

We also take a $\mathcal{P} \times_\alpha G$-tower $\{e_{i,j} |\ i,j \in \Lambda\}$ of the set X_0. We put

$$E_j = \mathrm{Dom}(e_{i,j}).$$

Corresponding to this tower, we choose a finite partition

$$\{E'_i |\ i \in \Lambda\}$$

of X'_0 such that

$$\mu'(E'_i |\ \theta(\omega)) = \mu(E_i |\ \omega) \qquad \text{a.e. } \omega.$$

We are going to show that for an arbitrary fixed index $i_0 \in \Lambda$ and for some measure preserving isomorphism ϕ: $E_{i_0} \to E'_{i_0}$, there exists a $\mathcal{P} \times_\alpha G$-tower

$\{e'_{i,j} |\, i,j \in \Lambda\}$ of the set X'_0 and an extended invertible measure preserving map $\phi: X_0 \to X'_0$ such that

$$\mathrm{Dom}(e'_{i,j}) = E'_j, \qquad \mathrm{Im}(e'_{i,j}) = E'_i,$$
$$\phi \cdot e_{i,j}(x) = e'_{i,j} \cdot \phi(x) \qquad (x \in E_j),$$
$$\alpha_g \cdot R^{-n} \cdot e_{i,j} \cdot \mathrm{Id}|_A \in [\mathcal{P}_\mu]_* \Leftrightarrow \alpha'_g \cdot R'^{-n} \cdot e'_{i,j} \cdot \mathrm{Id}|_{\phi(A)} \in [\mathcal{P}'_{\mu'}]_*$$

where $A \subset E_j$. We note that if $\alpha_g \cdot R^{-n} \cdot e_{i,j} \cdot \mathrm{Id}|_A \in [\mathcal{P}_\mu]_*$ then g and n are uniquely determined.

Each e_{j,i_0} is of the form:

$$e_{j,i_0} x = \alpha_g \cdot R^{-n} \cdot \gamma x \qquad (x \in E_{i_0}),$$

where $\gamma \in [\mathcal{P}_\mu]_*$ with $\mathrm{Dom}(\gamma) = E_{i_0}$, and $g = g(x,j) \in G, n = n(x,j) \in \mathbf{Z}$. As if necessary one can decompose each cell of the tower into an at most countable number of disjoint sets on which both $g(x,j)$ and $n(x,j)$ are constant, we may and do assume that $g(x,j)$ and $n(x,j)$ are functions of only j and write

$$g(j) = g(x,j), \quad n(j) = n(x,j) \quad (x \in E_{i_0}).$$

Since

$$\mu'(E'_{i_0} |\, \theta(\omega)) = \mu(E_{i_0} |\, \omega) \qquad \text{a.e. } \omega \in W^{\mathcal{P}_\mu},$$

we have a $\mu - \mu'$ preserving isomorphism $\phi: E_{i_0} \to E'_{i_0}$ such that

$$\mu'(\phi(A) |\, \theta(\omega)) = \mu(A |\, \omega) \quad \text{a.e. } \omega \in W^{\mathcal{P}_\mu} \qquad (\forall A \in E_{i_0} \cap \mathcal{B})$$

and such that

$$\phi(\zeta_{\mathcal{P}_\mu}(x) \cap E_{i_0}) = \theta(\zeta_{\mathcal{P}_\mu}(x)) \cap E'_{i_0},$$

where we consider $\zeta_{\mathcal{P}_\mu}(x)$ and $\theta(\zeta_{\mathcal{P}_\mu}(x))$ as subsets of X and X' respectively.

Then

$$\begin{aligned}
&\mu'(\alpha'_{g(j)} \cdot R'^{-n(j)}(E'_{i_0}) |\, \beta'_{[g(j)]} \cdot U'^{-n(j)} \cdot \theta(\omega)) \\
=&e^{f'(U'^{-n(j)} \cdot \theta(\omega), \theta(\omega))} \cdot \left(\frac{d\nu' U'^{-n(j)}}{d\nu'}(\theta(\omega)) \right)^{-1} \cdot \mu'(E'_{i_0} |\, \theta(\omega)) \\
=&e^{f(U^{-n(j)}\omega, \omega)} \cdot \left(\frac{d\nu U^{-n(j)}}{d\nu}(\omega) \right)^{-1} \cdot \mu(E_{i_0} |\, \omega) \\
=&e^{f(U^{-n(j)}\omega, \omega)} \cdot \left(\frac{d\nu U^{-n(j)}}{d\nu}(\omega) \right)^{-1} \cdot \mu(\gamma(E_{i_0}) |\, \omega)
\end{aligned}$$

(where $\gamma \in [\mathcal{P}_\mu]_*$ with $e_{j,i_0} = \alpha_{g(j)} \cdot R^{-n(j)} \cdot \gamma$)

$$
\begin{aligned}
&= \mu(\alpha_{g(j)} \cdot R^{-n(j)} \cdot \gamma(E_{i_0}) |\ \beta_{[g(j)]} \cdot U^{-n(j)}(\omega)) \\
&= \mu(e_j(E_{i_0}) | \beta_{[g(j)]} \cdot U^{-n(j)}(\omega)) \\
&= \mu(E_j |\ \beta_{[g(j)]} \cdot U^{-n(j)}(\omega)) \\
&= \mu'(E'_j |\ \beta'_{[g(j)]} \cdot U'^{-n(j)} \cdot \theta(\omega))
\end{aligned}
$$

where we use $\theta(\beta_{[g(j)]} \cdot U^{-n(j)}(\omega)) = \beta'_{[g(j)]} \cdot U'^{-n(j)}(\theta(\omega))$. So, using Hopf-equivalence by $\mathcal{P}'_{\mu'}$, we obtain $h'_j \in [\mathcal{P}'_{\mu'}]_*$ such that

$$
\begin{cases}
\mathrm{Dom}(h'_j) = \alpha'_{g(j)} \cdot R'^{-n(j)}(E'_{i_0}), \\
\mathrm{Im}(h'_j) = E'_j.
\end{cases}
$$

These partial transformations give us partial transformations $e'_{j,i_0}: E'_{i_0} \to E'_j$ by setting

$$
e'_{j,i_0} x' = h'_j \cdot \alpha'_{g(j)} \cdot R'^{-n(j)} x' \quad (x' \in E'_{i_0}).
$$

Then,

$$
\begin{cases}
e'_{j,i_0} \in [\mathcal{P}' \times_{\alpha'} G]_*, \\
\mathrm{Dom}(e'_{j,i_0}) = E'_{i_0}, \\
\mathrm{Im}(e'_{j,i_0}) = E'_j, \\
\zeta_{\mathcal{P}'_{\mu'}}(e'_{j,i_0} x) = \beta'_{[g(j)]} \cdot U'^{-n(j)}(\zeta_{\mathcal{P}'_{\mu'}}(x)), \\
\alpha_g \cdot R^{-n} \cdot e_{j,i_0} \cdot \mathrm{Id}|_A \in [\mathcal{P}_\mu]_* \Leftrightarrow \alpha'_g \cdot R'^{-n} \cdot e'_{j,i_0} \cdot \mathrm{Id}|_{\phi(A)} \in [\mathcal{P}'_{\mu'}]_*,
\end{cases}
$$

where $A \subset E_{i_0}$. We note that

$$
e_{j,i_0} \in [\mathcal{P} \times_\alpha H]_* \Leftrightarrow g(j) \in H \Leftrightarrow e'_{j,i_0} \in [\mathcal{P}' \times_{\alpha'} H]_*.
$$

Now let us extend ϕ to a $\mu - \mu'$ preserving measure isomorphism $X_0 \to X'_0$ by setting for each j

$$
\phi x = e'_{j,i_0} \cdot \phi \cdot e_{i_0,j} x \qquad (x \in E_j).
$$

Set

$$
\begin{cases}
e'_{i_0,j} = {e'_{j,i_0}}^{-1}, \\
e'_{j,l} = e'_{j,i_0} \cdot e'_{i_0,l}, \\
\xi' = \{e'_{j,l} |\ j, l \in \Lambda\}.
\end{cases}
$$

Thus we have constructed the desired $\mathcal{P}' \times_{\alpha'} G$-tower $\xi' = \{e'_{i,j} |\ i, j \in \Lambda\}$ of the set X'_0.

We take a $\mathcal{P}' \times_{\alpha'} G$-tower η' of the set E'_{i_0} and a product tower $\xi' \times \eta'$ of ξ' and η' so that the refinement $\xi' \times \eta'$ of ξ' approximates $\mathcal{S}'$-orbits and the measurable subsets of X'_0 in some fixed precision. Again take a corresponding partition of the set E_{i_0} and copy the tower η' into this set in the same way as the previous argument. Apply again this procedure and contiune back and forth in this fashion. In the limit we obtain a $\mu - \mu'$ preserving measure isomorphism $\phi: X_0 \to X'_0$ satisfying that for a.e. x,

$$\begin{cases} \phi(\mathcal{P} \times_\alpha G|_{X_0}(x)) = \mathcal{P}' \times_{\alpha'} G|_{X'_0}(\phi(x)), \\ \phi(\mathcal{P} \times_\alpha H|_{X_0}(x)) = \mathcal{P}' \times_{\alpha'} H|_{X'_0}(\phi(x)), \end{cases}$$

where $\mathcal{R}|_{X_0}$ means the restriction of the relation $\mathcal{R}$ to the set X_0. As $\mathcal{S}$ is of type III, we can find a countable parition $\{X_i |\ i \geq 0\}$ (resp. $\{X'_i |\ i \geq 0\}$) of X (resp. X') and partial transformations $d_i \in [\mathcal{S}]_*$ (resp. $d'_i \in [\mathcal{S}']_*$) such that $\mathrm{Dom}(d_i) = X_0$ and $\mathrm{Im}(d_i) = X_i$ (resp. $\mathrm{Dom}(d'_i) = X'_0$ and $\mathrm{Im}(d'_i) = X'_i$). Then the map $\phi: X_0 \to X'_0$ can be extended to an invertible map from X onto X' by setting

$$\phi(d_i x) = d'_i(\phi(x)), \qquad x \in X_0.$$

Then, obviously

$$\begin{cases} \phi(\mathcal{R}(x)) = \mathcal{R}'(\phi(x)), \\ \phi(\mathcal{S}(x)) = \mathcal{S}'(\phi(x)). \qquad \blacksquare \end{cases}$$

Remark 6.1: The part of the proof of Theorem 6.1 after Definition 6.2 shows us in general that if a countable amenable group G acts on a type III_0 amenable ergodic relation $\mathcal{P}$ as normalizers, then the pair of a normal subgroup $\{g \in G | \alpha_g \in [\mathcal{P}]\}$ and a conjugacy class of $\mathrm{mod}_{\tilde{\mathcal{P}}} \alpha_G$ (up to the commutant of the associated flow $F^{\mathcal{P}}$) is a complete invariant. This is known by S. I. Bezuglyi and V. Ya. Golodets in [BeGo]. Our proof using a copying lemma is very simple.

7. Splitting problem

Let $\mathcal{Q}$ be an ergodic relation of type III and $\{\mathcal{R}_0, \mathcal{S}_0\}$ be a pair of a type II ergodic relation and a subrelation. Taking the product relations $\mathcal{Q} \times \mathcal{R}_0$ and $\mathcal{Q} \times \mathcal{S}_0$, we obtain a type III inclusion $\mathcal{Q} \times \mathcal{S}_0 \subset \mathcal{Q} \times \mathcal{R}_0$, though. But apparently, this inclusion is essentially a type II inclusion. In this case we say that the inclusion is splitting. One can easily see that if $\mathcal{S} \subset \mathcal{R}$ is splitting then $F^{\mathcal{S}} = F^{\mathcal{R}}$. But the converse is not true.

PROPOSITION 7.1: *Suppose that* $[\mathcal{R}: \mathcal{S}] < \infty$ *and that* $\mathcal{R}$ *is amenable and of type* III_0. *Let* $\{\mathcal{P}, H \subset G, \alpha_G\}$ *be the canonical system for* $\mathcal{S} \subset \mathcal{R}$ *and* $G_0 = \{g \in G|\ \mathrm{mod}_{\tilde{\mathcal{P}}}\, \alpha_g = \mathrm{Id}\}$. *Then the following are all equivalent:*

1. *The inclusion* $\mathcal{S} \subset \mathcal{R}$ *is splitting.*
2. $F^{\mathcal{R}} = F^{\mathcal{P}}$.
3. $G_0 = G$.

Proof: The direction (1) $\Rightarrow$ (2) is obvious. (2) implies $|K| = 1$, i.e. (3) holds. The direction (3) $\Rightarrow$ (1) is not trivial. In fact, we need the assumption that $\mathcal{R}$ is amenable. In the proof of surjectivity of the map Υ in Theorem 6.1, we constructed a model having those complete invariants. If $G_0 = G$, then the relation and the subrelation $\mathcal{S} \subset \mathcal{R}$ constructed there is of the form $\mathcal{Q} \times \mathcal{S}_0 \subset \mathcal{Q} \times \mathcal{R}_0$, where $\mathcal{R}_0$ and $\mathcal{S}_0$ are of type II. Then, due to Theorem 6.1, we see this model is unique. Therefore (1) holds. ∎

8. Intermediate subrelation

By $\mathcal{A}$, we denote an intermediate subrelation between $\mathcal{R}$ and $\mathcal{S}$ which is defined by

$$\mathcal{P} \times_\alpha (H \vee G_0),$$

where $H \vee G_0$ means the group generated by the subgroup H and G_0 which is also a subgroup of G (see Sut[1]). We are concerned with the following two kinds of the inclusions $\mathcal{S} \subset \mathcal{R}$:

(1) $\mathcal{A} = \mathcal{S}$.

(2) $\mathcal{R} = \mathcal{A}$.

It is easily seen that

$$(1) \Leftrightarrow G_0 = \{e\}$$

and that

$$(2) \Leftrightarrow F^{\mathcal{R}} = F^{\mathcal{S}} \Leftrightarrow G = H \vee G_0 \Leftrightarrow K = L.$$

In this section we will investigate these two classes.

PROPOSITION 8.1: $\mathcal{A} = \mathcal{S}$ *if and only if the factor map* $\pi^{\mathcal{S}}_{\mathcal{R}}$ $(F^{\mathcal{R}}_t \pi^{\mathcal{S}}_{\mathcal{R}} = \pi^{\mathcal{S}}_{\mathcal{R}} F^{\mathcal{S}}_t)$ *is* $[\mathcal{R}: \mathcal{S}]$ *to* 1.

Proof: We note that $[\mathcal{R}\colon \mathcal{S}] = |G/H]$ and that $\pi^{\mathcal{S}}_{\mathcal{R}}$ is $|K/L|$ to 1. If $G_0 = \{e\}$, then $K = G, L = H$. Hence the factor map is $|G/H| = [\mathcal{R}\colon \mathcal{S}]$ to 1. Conversely if the factor map $\pi^{\mathcal{S}}_{\mathcal{R}}$ is $[\mathcal{R}\colon \mathcal{S}] = |G/H]$ to 1, then

$$|G/G_0| = |K| = |L| \cdot |K/L| = |L| \cdot |G/H| = |H/H_0| \cdot |G/H| = |G/H_0|$$

where $H_0 = H \cap G_0$. Hence, $H_0 = G_0$. On the other hand, the subgroup H does not contain a non-trivial normal subgroup of G. Therefore, $G_0 = \{e\}$. ∎

THEOREM 8.1 (Hamachi–Kosaki [HaKo2]): *Suppose $\mathcal{A} = \mathcal{S}$ and that $\mathcal{R}$ is amenable and of type III$_0$. Then the conjugacy class of a finite extension $(F^{\mathcal{S}}, F^{\mathcal{R}}, \pi^{\mathcal{S}}_{\mathcal{R}})$ is a complete invariant for the orbit equivalence of $\mathcal{S} \subset \mathcal{R}$.*

Proof: Since $K = G$ and $L = H$, by Theorem 2.4, $(F^{\mathcal{P}}, F^{\mathcal{R}}, \pi^{\mathcal{P}}_{\mathcal{R}}, L)$ is the minimal group cover of the extension $(F^{\mathcal{S}}, F^{\mathcal{R}}, \pi^{\mathcal{S}}_{\mathcal{R}})$. Therefore a conjugacy of $(F^{\mathcal{S}}, F^{\mathcal{R}}, \pi^{\mathcal{S}}_{\mathcal{R}})$ implies a conjugacy of the extension $(F^{\mathcal{P}}, F^{\mathcal{R}}, \pi^{\mathcal{P}}_{\mathcal{R}})$. By the main theorem we see that the conjugacy class of the extension $(F^{\mathcal{S}}, F^{\mathcal{R}}, \pi^{\mathcal{S}}_{\mathcal{R}})$ is a complete invariant. ∎

As for the case where $\mathcal{R} = \mathcal{A}$, since both the flow data from $\mathcal{R}$ and $\mathcal{S}$ respectively coincide, the inclusion might occur in a type II level. However, this is not true because of our criterion on splitting (Proposition 7.1). So therefore this type of inclusion will be interesting. Combining the fact $K = L$ and Theorem 6.1 we immediately have

PROPOSITION 8.2: *In the case $\mathcal{R} = \mathcal{A}$ and amenable type* III$_0$ *case, the extension $(F^{\mathcal{P}}, F^{\mathcal{S}}, \pi^{\mathcal{P}}_{\mathcal{S}})$ is an ergodic group K-extension and the collection of $((G, H, G_0), (F^{\mathcal{P}}, F^{\mathcal{S}}, \pi^{\mathcal{P}}_{\mathcal{S}}))$, where $G = H \vee G_0$, is a complete invariant for the inclusion $\mathcal{S} \subset \mathcal{R}$.*

In Proposition 3.1(b) of [Sut2] it is claimed that the triple (G, H, G_0) is a complete invariant in case of $\mathcal{R} = \mathcal{A}$. However, the flow data $(F^{\mathcal{P}}, F^{\mathcal{S}}, \pi^{\mathcal{P}}_{\mathcal{S}})$ are missing in the invariant which he obtained. As a matter of fact, the flow data in case of $\mathcal{R} = \mathcal{A}$ give us a lot of non-orbit equivalent inclusions as seen in the following remark.

Remark 8.1: By the construction of $\mathcal{S} \subset \mathcal{R}$ in the proof of the surjectivity of the map Υ in Theorem 6.1, if we have a family of non-conjugate group G/G_0 extension $\{(\{F^{\lambda}_t\}, \{S^{\lambda}_t\}, \pi^{\lambda}\}) \mid \lambda \in \Lambda\}$ where $G = H \vee G_0$ and $G \neq G_0$, then we obtain the corresponding family of $\mathcal{S}_\lambda \subset \mathcal{R}_\lambda$, $\lambda \in \Lambda$, which are mutually non-orbit equivalent and non-splitting. Examples of an uncountable family of non-conjugate G/G_0 extensions are known for instance in [Rud1] and [Rud2] in case of $G = S_3, H = S_2 \subset S_3$ and $G_0 = \mathbf{Z}_3 \subset S_3$.

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